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Lattices and Algorithms for Bivariate Bernstein, Lagrange, Newton, and Other Related Polynomial Bases Based on Duality between *L*-Bases and *B*-Bases

Suresh Kumar Lodha

Computer and Information Sciences, University of California, 241 Applied Sciences, Santa Cruz, California 95064
E-mail: lodha@cse.ucsc.edu

and

Ron Goldman

Department of Computer Science, Rice University, Houston, Texas 77251
E-mail: rng@cs.rice.edu

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L-Bases and *B*-bases are two important classes of polynomial bases used for representing surfaces in approximation theory and computer aided geometric design. It is well known that the Bernstein and multinomial (or Taylor) bases are special cases of both *L*-bases and *B*-bases. We establish that certain proper subclasses of bivariate Lagrange and Newton bases are *L*-bases. Furthermore, we present a rich collection of lattices (or point-line configurations) that admit unique Lagrange or Hermite interpolation problems which can be solved quite naturally in terms of Lagrange and Newton *L*-bases. A new geometric point-line duality between *L*-bases and *B*-bases is described: lines in *L*-bases correspond to points or vectors in *B*-bases and concurrent lines map to collinear points and vice versa. This duality between *L*-bases and *B*-bases is then used to establish that certain proper subclasses of power bases are *B*-bases and are dual to Lagrange *L*-bases. This geometric duality is further used to describe the lattices that admit power *B*-bases. *B*-bases dual to Newton *L*-bases are also investigated. Duality can also be used to develop change of basis algorithms with computational complexity $O(n^3)$ between any two *L*-bases and/or *B*-bases. We describe, in particular, a new change of basis algorithm from a bivariate Lagrange *L*-basis to a bivariate Bernstein basis with computational complexity $O(n^3)$.

1. INTRODUCTION

L-Bases and *B*-bases are local multivariate generalizations of Pólya bases and univariate *B*-splines [BGD91, CM92, DMS92]. These two important classes of polynomial bases have been used for representing surfaces in

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computer aided geometric design. B-bases are blending functions for B-patches, which were first introduced by Seidel via blossoming [Sei91] and then shown to agree with certain multivariate B-splines on a special region of the parameter domain [DMS92]. L-Bases consist of polynomials that can be factored in a special way into products of linear functions. These bases were first studied by Cavaretta and Micchelli, who showed that L-bases are dual to B-bases using a multivariate polynomial identity [CM92]. Similar results on the algebraic duality between B-bases and L-bases were obtained by the authors by generalizing the de Boor–Fix dual functionals [dBF73] from curves to surfaces [LG94b].

The purpose of this paper is to flesh out the theory of L-bases and B-bases with examples of important special cases. It is well-known that the Bernstein and multinomial (or Taylor) bases are special cases of both L-bases and B-bases. It is therefore natural to ask whether any other well-known bivariate bases can be realized as special cases of L-bases or B-bases. Lagrange, Newton, and Hermite bases for surfaces are particularly important in approximation theory and computer aided design because these bases are known to be very well suited for interpolating point and derivative data. The literature on this subject is huge, continuing, and growing and we refer the reader to only a few representative references [Gas90, M90, GM89, dBR92, dBR90, LL90, Lor90, Muh80, Muh74, NR92].

Here we shall establish for the first time that certain proper subclasses of Lagrange and Newton bivariate bases arise as L-bases. We shall also exhibit a rich collection of lattices or point-line configurations that admit unique solutions to certain point and derivative interpolation problems by bivariate Lagrange and Newton L-bases. Specifically, we shall prove that every principal lattice or geometric mesh [CY77] admits unique interpolation by Lagrange and Newton L-bases. Moreover, we shall show that every natural lattice [CY77] and certain other lattices usually associated with Hermite interpolation problems also admit unique interpolation by Newton L-bases.

We then proceed to further refine the theory of duality between L-bases and B-bases. It turns out that bivariate Bernstein–Bézier and multinomial bases are dual to bivariate Bernstein–Bézier and multinomial bases. But what are the dual bases to the bivariate Lagrange and Newton L-bases? To answer this question, we first introduce the power and Newton dual bases for surfaces and demonstrate that certain subclasses of these bases arise as B-bases—that is, bases that are blending functions for B-patches [Sei91]. We then establish that the Lagrange and generalized Newton bases are dual to the power and generalized Newton dual bases.

L-Bases are represented by lattices of lines, B-bases by lattices of points and vectors. In order to portray the lattices that admit B-bases and, in particular, power B-bases, we introduce a geometric principle of duality between representations for B-bases and representations for L-bases under which lines representing L-bases correspond to points or vectors representing B-bases and concurrent lines map to collinear points. This geometric duality is a powerful tool leading to a better intuitive understanding of the lattices that represent B-bases and L-bases.

Change of basis algorithms of $O(n^3)$ for degree n bivariate B-bases have been developed based on blossoming [LG95a]. Similarly, $O(n^3)$ change of basis algorithms for L-bases have been constructed using duality [LG95a]. Our rich collection of examples of B-bases and L-bases along with the duality principle unifies within a single framework a large variety of bivariate polynomial bases including Lagrange, Newton, power, Bernstein–Bézier, multinomial, and Newton dual bases. This unification yields an elegant change of basis algorithm between any two of these bases with computational complexity $O(n^3)$. Here we shall present for the first time the special case of the change of basis algorithm between the Lagrange and Bernstein–Bézier L-bases. The inverse transformation from Bernstein–Bézier form to Lagrange form yields an evaluation algorithm for degree n Bernstein–Bézier surfaces with an amortized cost of $O(n)$ computations per point [LG95c].

Our work easily generalizes to higher dimensions. Nevertheless, for the sake of simplicity, the results are presented and derived here only for surfaces.

Many of our results are bivariate generalizations of known univariate results, but with important differences. These differences arise from the nature of the theory of multivariate polynomial interpolation, which is well known to be inherently more complicated than the theory of univariate polynomial interpolation both because of the wide variety of subspaces of multivariate polynomials one can choose, and because the solvability of the interpolation problem depends on the geometric distribution of the points. We have discovered that another important reason for these differences is that there is no meaningful analogue of geometric duality in the univariate setting.

We now briefly describe some of the major differences between the univariate and bivariate theories which are discussed further in this work. First, Goldman and Barry established in the univariate case that *all* Lagrange and Newton bases can be realized as Pólya bases [GB92]. In contrast, we demonstrate that only a strict subclass of bivariate Lagrange and Newton bases can be realized as L-bases. Nevertheless, by presenting examples of lattices, we also establish that this strict subclass is fairly rich and interesting. Second, the algebraic duality between B-splines and Pólya bases can be derived from Marsden's identity [Mar70] or from dual functionals using blossoming techniques [BGD91, Ram87, Ram89, Ram88, Sei93]. Relatively recently algebraic duality between B-bases and L-bases has been demonstrated by generalizing the Marsden identity [Mar70] to the multivariate setting [CM92] and also by extending the de Boor–Fix formula to the multivariate case using blossoming techniques [LG94b]. However, geometric point-line duality has not been previously observed. This geometric point-line duality turns out to be very useful for gaining further insight into the structure of lattices as well as for deriving associated formulas and algorithms. Geometric point-line duality yields a new, simpler, and more elegant form of the multivariate de Boor–Fix formula, that does not need the alternating sums arising in the univariate de Boor–Fix formula [LG94b]. Moreover, the fact that certain proper subclasses of power bases—bases consisting of nth powers of linear polynomials (i.e., lines)—can be realized as B-bases, which are defined in terms of points, is quite intriguing. A very simple proof of this fact, presented in Section 4.2.3, can be obtained from duality; however a direct proof presented in the Appendix is interesting in its own right. Third, Goldman and Barry have studied change of basis algorithms between univariate B-splines and Pólya bases with computational complexity $O(n^2)$ [GB92]. In our earlier work [LG95a], we extended these results to derive change of basis algorithms between L-bases and B-bases with computational complexity $O(n^3)$. Since here we shall establish that certain bivariate Lagrange and Newton bases can be realized as L-bases, we can now derive new change of basis algorithms with computational complexity $O(n^3)$ between the bivariate Lagrange L-bases, Newton L-bases, Bernstein–Bézier bases, and multinomial bases; these algorithms are described in Section 5. To the best of our knowledge, change of basis algorithms with computational complexity $O(n^3)$ between bivariate Lagrange bases and Bernstein–Bézier bases have not been reported in the literature. Therefore, we believe that the algorithm presented in this work is of both theoretical and practical interest. Finally, in hindsight, most of the results in this work may seem very natural extensions of similar results from the univariate setting, and therefore are, in fact, very satisfying. However, extensions from the univariate case to the multivariate case, if they do exist, are far from trivial. To reinforce this point, we raise an important related open question, which again rests on some fundamental differences between the univariate and multivariate cases. Goldman [Gol94] has recently extended duality, blossoming techniques, and recursive algorithms to arbitrary univariate polynomial bases. These techniques have also been extended to piecewise polynomial spaces defined by connection matrices [BDGM91, BGM93]. But it is still an open question whether these techniques can be extended to arbitrary bivariate polynomial bases or even to the bivariate polynomial bases that factor into linear factors.

The remainder of this paper is organized in the following manner. Section 2 reviews the definitions of L-bases and B-bases. Section 3 focuses on duality: A geometric point-line duality is introduced between representations for B-bases and L-bases, and an algebraic duality is formulated from a generalization of the de Boor–Fix formula from curves to surfaces. Many interesting examples of dual bases are provided in Section 4 including the general bivariate Bernstein–Bézier and multinomial bases, special Lagrange and power bases, and certain Newton and Newton dual bases. In Section 5 we turn our attention to algebraic duality. Here we mention various dual formulas and algorithms based on the algebraic duality between B-bases and L-bases that arise from the generalized de Boor–Fix formula. We focus, in particular, on change of basis algorithms for L-bases, and we exhibit these procedures by converting a bivariate polynomial from Lagrange to Bernstein–Bézier form. We conclude in Section 6 with a short summary of our work and a brief discussion of future research.

Throughout this paper, we shall adopt the following notation. A multi-index α is a 3-tuple of nonnegative integers. If $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, then $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\alpha_1!\,\alpha_2!\,\alpha_3!$. Other multi-indices will be denoted by β and γ. A unit multi-index e_k is a 3-tuple with 1 in the kth position and 0 everywhere else. Scalar indices will be denoted by i, j, k, l. Finally, given a homogeneous polynomial $f(x, y, z)$, $D^\alpha f$ denotes $\partial^{|\alpha|}f/\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}$.

2. BASES

Here we review the basic definitions and certain well-known properties of homogeneous and affine L-bases and B-bases. We also provide geometric interpretations for the algebraic entities associated with these bases.

2.1. *L-Bases*

A collection $\mathscr{L}$ of three sequences $\{L_{1,j}\}$, $\{L_{2,j}\}$, $\{L_{3,j}\}$, $j=1, ..., n$ of linear homogeneous (resp. affine) polynomials in three (resp. two) variables is called a *knot-net* of homogeneous (resp. affine) polynomials if $(L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1})$ are linearly (resp. affinely) independent polynomials for $0 \leqslant |\alpha| \leqslant n-1$. A homogeneous (resp. affine) L-basis $\{l^n_\alpha, |\alpha|=n\}$ is a collection of $\binom{n+2}{2}$ trivariate (resp. bivariate) polynomials defined as follows:

$$l^n_\alpha=\prod_{i=1}^{\alpha_1} L_{1i} \prod_{j=1}^{\alpha_2} L_{2j} \prod_{k=1}^{\alpha_3} L_{3k}. \tag{1}$$

It is well known that $\{l^n_\alpha, |\alpha|=n\}$ is, in fact, a homogeneous (resp. affine) basis for the space of homogeneous (resp. affine) polynomials of degree n on R^3 (resp. R^2) [CM92].

By associating the homogeneous polynomial $L = ax + by + cz$ with the affine polynomial $A = ax + by + c$, one can define a one-to-one correspondence between the knot-net of homogeneous and affine polynomials and between the homogeneous and affine L-bases. Due to this one-to-one correspondence between homogeneous and affine L-bases, in the following discussions we shall refer to either the homogeneous or affine L-basis, whichever is more convenient or intuitive in the particular context.

Furthermore, we assign to each homogeneous (resp. affine) polynomial, the following geometric interpretation. The polynomial $ax + by + cz$ (resp. $ax + by + c$) corresponds to the line in the projective (resp. affine) plane defined by the equation $ax + by + cz = 0$ (resp $ax + by + c = 0$). In particular, the polynomial cz (resp. c) corresponds to the line at infinity in the projective plane. Observe that this correspondence between the lines and polynomials depends on the coordinate system and is unique only up to constant multiples. As a result, polynomials that differ by constant multiples map to the same line in the projective plane. Nevertheless, we shall identify the polynomial with the line and vice versa in the following discussions, whenever the coordinate system and constant multiples are irrelevant for the context at hand. Under this mapping, a set of three polynomials is linearly independent iff the corresponding lines are not concurrent—that is, the three lines do not pass through a common point in the projective plane. The advantage of this correspondence is to allow us to think of algebraic entities such as polynomials in terms of geometric entities such as lines.

There are many models of the projective plane such as the sphere model and the plane model [Sto89]. In this work, we will think of the projective plane as the the real plane R^2 together with a line at infinity. This realization of the projective plane is shown on the left-hand side of Fig. 1. A solid arrows indicates that the correspondence is many to one; a double arrow indicates a 1–1 correspondence.

2.2. *B-bases*

A collection $\mathscr{U}$ of three sequences $\{\mathbf{u}_{1,j}\}$, $\{\mathbf{u}_{2,j}\}$, $\{\mathbf{u}_{3,j}\}$, $j = 1, ..., n$ of vectors in R^3 is called a *knot-net* of vectors if $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ are linearly independent vectors in R^3 for $0 \leqslant |\alpha| \leqslant n-1$. One can write any vector $\mathbf{u}$ in R^3 in terms of the basis $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ so that

$$\mathbf{u} = \sum_{k=1}^{3} h_{k,\alpha}(\mathbf{u})\, \mathbf{u}_{k,\alpha_k+1}.$$

Notice that the functions $h_{k,\alpha}$ are trivariate homogeneous polynomials.

A homogeneous B-patch of degree n over the knot-net $\mathscr{U}$ is a trivariate homogeneous polynomial B: $R^3 \to R^m$ defined by the following recurrence.

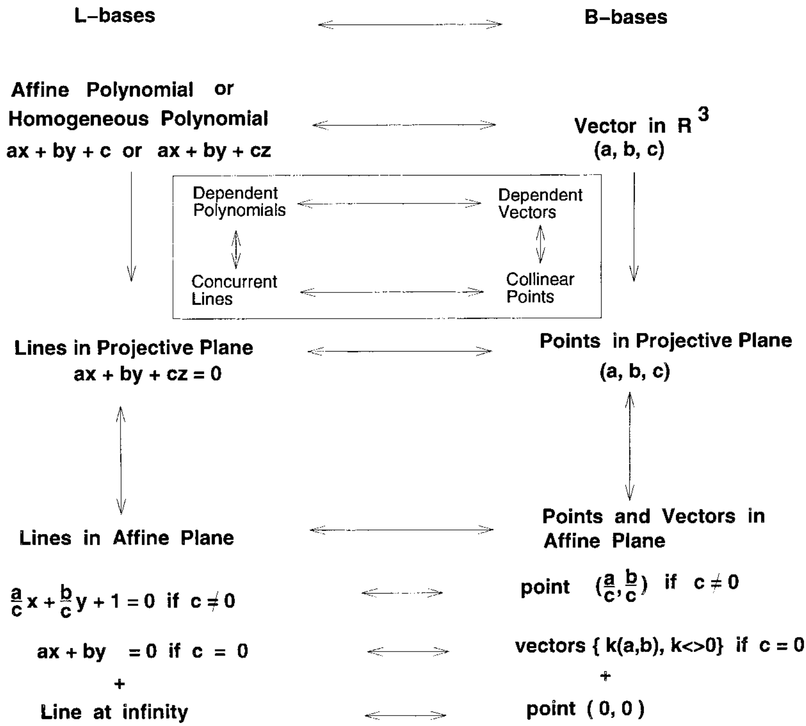

Fig. 1. Point-line duality.

The initial conditions for the recurrence are given by setting $C^0_\alpha(\mathbf{u}) = C_\alpha \in R^m$ for $|\alpha| = n$. The recurrence is constructed for $|\alpha| = n - l$, $l = 1, ..., n$ by

$$c^l_\alpha(\mathbf{u}) = \sum_{k=1}^{3} h_{k,\alpha}(\mathbf{u})\, C^{l-1}_{\alpha+e_k}(\mathbf{u}). \tag{2}$$

The homogeneous B-patch is then defined as $B(\mathbf{u}) = C^n_0(\mathbf{u})$. This algorithm is known as the *up* recurrence; it generalizes to surfaces the de Boor evaluation algorithm for B-spline curves [dB72]. A homogeneous B-basis $\{b^n_\alpha, |\alpha| = n\}$ is a collection of $\binom{n+2}{2}$ homogeneous trivariate polynomials from R^3 to R defined by choosing the constants $C_\beta \in R$ as follows:

$$\begin{aligned} C_\beta &= 1 \qquad \text{if} \quad \beta = \alpha \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

It has been shown that $\{b^n_\alpha, |\alpha| = n\}$ is, in fact, a basis for the space of homogeneous polynomials on R^3 [Sei91]. Moreover, an arbitrary homogeneous B-patch of degree n can be represented in terms of a homogeneous B-basis as follows:

$$B(\mathbf{u}) = \sum_{|\alpha| = n} C_\alpha b_\alpha^n(\mathbf{u}).$$

We can map vectors in R^3 to points in the projective plane, by assigning to each vector $\mathbf{u} = (a, b, c)$ the corresponding point $\mathbf{v} = (a, b, c)$ in the projective plane. More explicitly, to a vector $\mathbf{u} = (a, b, c)$ in R^3, we shall associate the affine point $\mathbf{v} = (a/c, b/c, 1)$ in the projective plane whenever $c \neq 0$ and the point at infinity $\mathbf{v} = (a, b, 0)$ in the projective plane whenever $c = 0$. Observe that this correspondence between the vectors in R^3 and points in the projective plane depends on the coordinate system and is unique only up to constant multiples. As a result, vectors that differ by constant multiples map to the same points in the projective plane. Nevertheless, we shall identify the vector with the point and vice versa, whenever the coordinate system and constant multiples are irrelevant for the context at hand. This correspondence is shown on the right-hand side of Fig. 1.

Under this mapping, a set of three vectors in R^3 is linearly independent iff the corresponding point are not collinear—that is, the three points do not lie on a straight line in the projective plane. The advantage of this correspondence is to allow us to think of vectors in R^3 in terms of points in the plane. In Section 3.2 we shall invoke this model to construct a geometric point-line duality between knot-nets for L-bases (lines) and knot-nets for B-bases (points). We shall then apply this model in Sections 4, 5 to bolster our intuition and to develop lattices and algorithms for B-bases and L-bases.

As noted in Section 2.1, in this work we model the projective plane as the real plane R^2 together with a line at infinity. This realization of the projective plane is somewhat more interesting for B-bases than for L-bases because for B-bases we also want to think of points at infinity as "vectors" or "directions" in R^2. Specifically, we will think of the point at infinity on the line $bx - ay = 0$ in the projective plane—that is the point at infinity in the direction of the vector (a, b) or $(-a, -b)$—as the "vector" in the direction (a, b) up to constant multiples. More formally, a point at infinity in the direction (a, b) represents the equivalence class of vectors $k(a, b)$ where k is any non-zero constant. Rather than referring to these equivalence classes of vectors, we shall simply refer to these classes as vectors, whenever the constant multiples are irrelevant for the context at hand. This correspondence is shown on the lower right-hand side of Fig. 1. The distinction between points and vectors in R^2 will be emphasized only when it is relevant to the context. It is remarkable that this distinction vanishes after homogenization and that homogenization holds the key to dealing with point and derivative information on equal footing.

Under this realization of the projective plane, it is instructive to write down explicitly what is meant by the linear independence of points and vectors in R^2. Given any three points or vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in R^2, there are three distinct cases to consider:

1. $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are all points. Three points in R^2 are said to be linearly independent iff they are not collinear or alternatively iff they form a non-degenerate triangle.

2. Two of the three, say $\mathbf{v}_1$ and $\mathbf{v}_2$, are points and the third one $\mathbf{v}_3$ is a vector. These entities are said to be linearly independent iff $\mathbf{v}_1 \neq \mathbf{v}_2$ and the vector $\mathbf{v}_3$ does not lie along the straight line determined by the two points $\mathbf{v}_1$ and $\mathbf{v}_2$.

3. Two of the three, say $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors and the third one $\mathbf{v}_3$ is a point. These entities are said to be linearly independent iff the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent in R^2.

The fourth and only remaining case when $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are all vectors is not of interest to us because three vectors in R^2 are always linearly dependent. Now it is easy to verify that with the correspondence between points and vectors of R^2 and vectors of R^3, three points or vectors in R^2 are linearly independent iff the corresponding vectors in R^3 are linearly independent.

3. DUALITY BETWEEN *B*-BASES AND *L*-BASES

We now describe the duality between *B*-bases and *L*-bases from two different perspectives: algebraic and geometric.

3.1. *de Boor–Fix Duality*

Given a knot-net of vectors $\mathbf{u}_{i,j}$ in R^3, consider the knot-net of linear homogeneous polynomials $L_{i,j}$ defined by the correspondence:

$$(a, b, c) \leftrightarrow (ax + by + cz).$$

Let l_α^n be the *L*-basis functions defined by the knot-net $L_{i,j}$, and let b_β^n be the *B*-basis functions defined by the knot-net $\mathbf{u}_{i,j}$.

The bases l_α^b and b_β^n are related algebraically through the following bilinear form, also referred to as the bracket operator. Given any two homogeneous polynomials $f, g\colon R^3 \to R$ of degree n, define the bilinear form

$$[f, g](\mathbf{u}) = \frac{1}{n!} \sum_{|\alpha|=n} \frac{D^\alpha f(\mathbf{u}) * D^\alpha g(\mathbf{u})}{\alpha!}.$$

Note that this bracket operator depends on n, and therefore, strictly speaking, the notation $[f, g]_n$ is more appropriate. However, we shall suppress the subscript n, whenever it does not cause any ambiguity.

THEOREM 1. *Generalized de Boor–Fiz formula* [LG94b]: $[l^n_\alpha, b^n_\beta] = \delta_{\alpha\beta}$.

COROLLARY 1. *Cavaretta–Micchelli identity* [CM92, LG95a]: $\sum_{|\alpha|=n} l^n_\alpha(x, y, z)\, b^n_\alpha(a, b, c) = (ax + by + cz)^n$.

Because of Theorem 1, the L-basis l^n_α can be used to represent the dual functionals for the B-basis b^n_β and vice versa. We shall explore some of the consequences of this algebraic duality in Section 5.

3.2. *Point-Line Duality*

The correspondence $(a, b, c) \leftrightarrow ax + by + cz = (a, b, c) \cdot (x, y, z)$ associates to each vector in R^3 a homogeneous trivariate polynomial. Earlier we saw that vectors in R^3 correspond to points in the projective plane (or points and vectors in the affine plane), and homogeneous trivariate polynomials correspond to lines in the projective plane (or lines in the affine plane plus the line at infinity). Thus B-bases are represented by knot-nets of points $\mathbf{u}_{ij}$ in the projective plane and L-bases by knot-nets of lines L_{ij} in the projective plane. We say that a B-basis and an L-basis are dual bases if their knot-nets are related by the correspondence $L_{ij} = \mathbf{u}_{ij} \cdot (x, y, z)$. Under this correspondence points in the projective plane are mapped to lines in the projective plane and collinear points are mapped to concurrent lines.

Figure 1 summarizes the relationships between dual B-bases and L-bases, as well as the algebra and geometry underlying their associated knot-nets. The duality between L-bases and B-bases is indicated by double-sided arrows between the left and right parts of Fig. 1. The double-sided arrow indicates a 1–1 correspondence.

4. EXAMPLES OF DUAL BASES

In this section we discuss three sets of examples of dual B-bases and L-bases: dual Bernstein–Bézier and multinomial bases, dual Lagrange and power bases, and dual Newton and Newton dual bases. We begin by showing how each of these bases can be realized as a B-basis or an L-basis by constructing the appropriate knot-nets. We go on to discuss the geometry of these knot-nets as well as the geometry of the knot-nets for the corresponding dual bases. Later we shall see that while the correspondence at the homogenized level is simpler and more elegant algebraically, the point-line correspondence at the affine level provides better geometric insight.

4.1. *Duality between Bernstein–Bézier and Multinomial Bases*

This section explains how to realize any Bernstein–Bézier or multinomial basis as a special case of both B-bases and L-bases. The fact that these bases can be realized as both B-bases and L-bases is not new. However, the duality between these bases is new. We present this case in some detail not only for the sake of completeness, but also because the geometric duality between these bases and their associated knot-nets is simple to understand and enhances both the comprehension and appreciation of the duality between other L-bases and B-bases to be discussed later in this section. To help investigate the duality between bivariate Bernstein–Bézier and multinomial bases, we also introduce the hybrid Bernstein–Bézier Multinomial (BM) basis. We shall refer to a B-basis as a *uniform* B-basis if the knot-net $\mathbf{u}_{ij}$ satisfies the property: $\mathbf{u}_{ij} = \mathbf{u}_i$ for $j = 1, ..., n$. A uniform L-basis is defined in an analogous manner.

4.1.1. *Bernstein–Bézier Bases*

First we describe how Bernstein–Bézier bases can be realized as special cases of B-bases. Let $\mathbf{u}_1 = (a_1, b_1, c_1)$, $\mathbf{u}_2 = (a_2, b_2, c_2)$, and $\mathbf{u}_3 = (a_3, b_3, c_3)$ be three linearly independent vectors in R^3 such that $c_i \neq 0$ for $i = 1, 2, 3$. Choose the uniform knot-net of vectors $\mathbf{u}_{i,j} = \mathbf{u}_i$, $1 \leqslant j \leqslant n$. Then the corresponding B-basis is a homogeneous Bernstein–Bézier basis. For example, if $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 1)$ and $\mathbf{u}_3 = (0, 0, 1)$, then it is easy to verify that the B-basis functions are also the homogeneous Bernstein–Bézier basis functions; that is

$$b^n_\alpha(x, y, z) = \frac{n!}{\alpha!} x^{\alpha_1} y^{\alpha_2} (z - x - y)^{\alpha_3}.$$

More generally, if $\mathbf{u}_1 = (a_1, b_1, 1)$, $\mathbf{u}_2 = (a_2, b_2, 1)$, and $\mathbf{u}_3 = (a_3, b_3, 1)$, then it can readily be verified that the B-basis functions are indeed homogeneous Bernstein–Bézier basis functions; that is,

$$b^n_\alpha(x, y, z) = \frac{n!}{\alpha!} h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3} z^n,$$

where (h_1, h_2, h_3) are the barycentric coordinates of the point $(x/z, y/z)$ with respect to the points (a_1, b_1), (a_2, b_2), and (a_3, b_3). Even more generally, if $c_i \neq 0$ for $i = 1, 2, 3$ and $\mathbf{u}_1 = (a_1, b_1, c_1)$, $\mathbf{u}_2 = (a_2, b_2, c_2)$, and $\mathbf{u}_3 = (a_3, b_3, c_3)$, then it can be verified that the B-basis functions are again homogeneous Bernstein–Bézier basis functions; this time,

$$b_\alpha^n(x, y, z) = \frac{n!}{\alpha!}\left(\frac{h_1}{c_1}\right)^{\alpha_1}\left(\frac{h_2}{c_2}\right)^{\alpha_2}\left(\frac{h_3}{c_3}\right)^{\alpha_3} z^n,$$

where (h_1, h_2, h_3) are the barycentric coordinates of the point $(x/z, y/z)$ with respect to the points $(a_1/c_1, b_1/c_1)$, $(a_2/c_2, b_2/c_2)$, and $(a_3/c_3, b_3/c_3)$.

We can also realize Bernstein–Bézier bases as special cases of L-bases. Let $L_1 = a_1x + b_1y + c_1z$, $L_2 = a_2x + b_2y + c_2z$, and $L_3 = a_3x + b_3y + c_3z$ be three linearly independent polynomials. Furthermore, assume that the following three conditions are satisfied: $a_1b_2 - a_2b_1 \neq 0$, $a_2b_3 - a_3b_2 \neq 0$, and $a_3b_1 - a_1b_3 \neq 0$, that is, no two of the associated lines are parallel. Choose the uniform knot-net of polynomials $L_{i,j} = L_i$, $1 \leqslant j \leqslant n$. Then the corresponding L-basis is a homogeneous Bernstein–Bézier basis up to constant multiples. Indeed, one can easily verify that up to constant multiples this L-basis is the homogenized Bernstein–Bézier basis defined by the three intersection points of L_1, L_2, and L_3: $\mathbf{u}_1 = (b_2c_3 - b_3c_2, a_3c_2 - a_2c_3, a_2b_3 - a_3b_2)$, $\mathbf{u}_2 = (b_3c_1 - b_1c_3, a_1c_3 - a_3c_1, a_3b_1 - a_1b_3)$, and $\mathbf{u}_3 = (b_1c_2 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1)$. In fact with this choice of points, the linear L-basis, which is the same as the barycentric coordinates with respect to the triangle defined by these three points, is precisely $((a_2b_3 - a_3b_2)/\Delta)\, L_1(\mathbf{u})$, $((a_3b_1 - a_1b_3)/\Delta)\, L_2(\mathbf{u})$, and $((a_1b_2 - a_2b_1)/\Delta)\, L_3(\mathbf{u})$ or alternatively, $L_1(\mathbf{u})/L_1(\mathbf{u}_1)$, $L_2(\mathbf{u})/L_2(\mathbf{u}_2)$, and $L_3(\mathbf{u})/L_3(\mathbf{u}_3))$ where Δ is the determinant of the matrix defined by (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3). In particular, $L_1 = x$, $L_2 = y$, and $L_3 = -x - y + z$ yields the standard homogeneous Bernstein–Bézier basis, up to constant multiples, that is, $l_\alpha^n = x^{\alpha_1}y^{\alpha_2}(z - x - y)^{\alpha_3}$. In summary, given a triangle, we can use the vertices to define the Bernstein–Bézier basis—this is the B-basis point of view—or we can use the lines to define the Bernstein–Bézier basis—this is the L-basis point of view.

4.1.2. *Multinomial Bases*

The multinomial basis is the standard generalization of the monomial basis to the multivariate setting. For example, the basis $1, x, y, x^2, xy$, and y^2 is the bivariate multinomial basis of degree 2. Sometimes the terminology Taylor basis or power basis is also used instead of monomial or multinomial basis. However, we shall refer to this basis as the multinomial basis in accordance with [GB92] and reserve the term power basis for the basis discussed later in Section 4.2.2.

We first describe how to realize multinomial bases as special cases of B-bases. Let $\mathbf{u}_1=(a_1, b_1, c_1)$, $\mathbf{u}_2=(a_2, b_2, c_2)$, and $\mathbf{u}_3=(a_3, b_3, c_3)$ be three linearly independent vectors in R^3 such that $c_1=c_2=0$. Observe that by the linear independence condition $c_3 \neq 0$. Choose the uniform knot-net of vectors $\mathbf{u}_{i,j}=\mathbf{u}_i$, $1 \leqslant j \leqslant n$. Then the corresponding B-basis is a homogeneous multinomial basis up to constant multiples. In other words, the multinomial basis is defined by a point and two linearly independent vectors in R^2. The simplest and most popular example of this construction is obtained by setting $\mathbf{u}_1=(1, 0, 0)$, $\mathbf{u}_2=(0, 1, 0)$, and $\mathbf{u}_3=(0, 0, 1)$. In this case it is easy to verify that the B-basis functions are homogeneous multinomial basis functions, and that

$$b^n_\alpha(x, y, z)=\frac{n!}{\alpha!}\, x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}.$$

If a homogeneous polynomial $B(\mathbf{u})$ has coefficients C_α with respect to the standard multinomial B-basis, then

$$B(\mathbf{u})=\sum_{|\alpha|=n} \frac{n!}{\alpha!}\, C_\alpha x^{\alpha_1} y^{\alpha_2} z^{\alpha_3},$$

and the coefficients C_α represent, up to constant multiples, the directional derivatives of the polynomial $B(\mathbf{u})$ at the point $(0, 0)$ along the directions $(1, 0)$ and $(0, 1)$. The multinomial basis defined by a point $\mathbf{v}_1$ and two vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ is a generalization where the coefficients of a polynomial with respect to this multinomial basis represent, up to constant multiples, the directional derivatives of this polynomial at the point $\mathbf{v}_1$ along the directions $\mathbf{v}_2$ and $\mathbf{v}_3$. As an example, if $\mathbf{u}_1=(1, -1, 0)$, $\mathbf{u}_2=(1, 1, 0)$, and $\mathbf{u}_3=(0, 0, 1)$, then it can readily be verified that the B-basis functions are again homogeneous multinomial basis functions; that is,

$$b^n_\alpha(z, y, z)=\frac{n!}{\alpha!}\left(\frac{x-y}{2}\right)^{\alpha_1}\left(\frac{x+y}{2}\right)^{\alpha_2} z^{\alpha_3},$$

where $(x+y)/2$ and $(x-y)/2$ represent the directions $(1, -1)$ and $(1, 1)$ along which the multinomial basis is formed instead of along the usual directions $(1, 0)$ and $(0, 1)$. As another example, if $\mathbf{u}_1=(1, 0, 0)$, $\mathbf{u}_2=(0, 1, 0)$, and $\mathbf{u}_3=(a_3, b_3, 1)$, then it can readily be verified that the B-basis functions are indeed homogeneous multinomial basis functions; this time,

$$b^n_\alpha(x, y, z)=\frac{n!}{\alpha!}\,(x-a_3 z)^{\alpha_1}\,(y-b_3 z)^{\alpha_2}\, z^{\alpha_3},$$

where the multinomial basis is formed at (a_3, b_3) along the usual directions $(1, 0)$ and $(0, 1)$. More generally, if $\mathbf{u}_1=(a_1, b_1, 0)$, $\mathbf{u}_2=(a_2, b_2, 0)$, and $\mathbf{u}_3=(a_3, b_3, 1)$, then the homogeneous B-basis functions are

$$b^n_\alpha(x, y, z)=\frac{n!}{\alpha!}\left(\frac{b_2(x-a_3z)-a_2(y-b_3z)}{a_1b_2-a_2b_1}\right)^{\alpha_1} \times\left(\frac{-b_1(x-a_3z)+a_1(y-b_3z)}{a_1b_2-a_2b_1}\right)^{\alpha_2} z^{\alpha_3},$$

where the multinomial basis is formed at (a_3, b_3) along the directions (a_1, b_1) and (a_2, b_2).

We can also realize multinomial bases as special cases of L-bases. Let $L_1=a_1x+b_1y+c_1z$, $L_2=a_2x+b_2y+c_2z$, and $L_3=z$ be three linearly independent polynomials. Observe that by the linear independence condition, it follows that $a_1b_2-a_2b_1\neq 0$; thus the lines corresponding to L_1 and L_2 are not parallel. Choose the uniform knot-net of polynomials $L_{i,j}=L_i$, $1\leqslant j\leqslant n$. Then one can easily verify that this L-basis is indeed the homogenized multinomial basis defined by the vectors $(b_2/k, -a_2/k)$ and $(-b_1/k, a_1/k)$ and the point $((b_1c_2-b_2c_1)/(a_1b_2-a_2b_1), (c_1a_2-c_2a_1)/(a_1b_2-a_2b_1))$, where $k=a_1b_2-a_2b_1$. In particular choosing $L_1=x$, $L_2=y$, and $L_3=z$, yields the standard homogeneous multinomial basis; that is $l^n_\alpha=x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$. Also choosing $L_1=x-az$, $L_2=y-bz$, and $L_3=z$, yields the homogeneous multinomial basis defined by the point (a, b) and the unit vectors $(1, 0)$ and $(0, 1)$; that is $l^n_\alpha=(x-az)^{\alpha_1}(y-bz)^{\alpha_2}z^{\alpha_3}$.

4.1.3. *Hybrid Bernstein–Bézier Multinomial (BM) Bases*

We now introduce hybrid Bernstein–Bézier Multinomial (BM) bases in order to help describe the duality between Bernstein–Bézier and multinomial bases in the next Section 4.1.4. A Bernstein–Bézier B-basis is defined by three points, while a multinomial B-basis is defined by a point and two vectors. A hybrid Bernstein–Bézier Multinomial basis is defined by two points and a vector.

Any hybrid BM basis can be realized as a B-basis as follows: Let $\mathbf{u}_1=(a_1, b_1, c_1)$, $\mathbf{u}_2=(a_2, b_2, c_2)$, and $\mathbf{u}_3=(a_3, b_3, c_3)$ be three linearly independent vectors in R^3 such that $c_1=0$, $c_2\neq 0$, and $c_3\neq 0$. Choose the uniform knot-net of vectors $\mathbf{u}_{i,j}=u_i$, $1\leqslant j\leqslant n$. The corresponding B-basis will be referred to as a hybrid homogeneous BM basis. This basis is formed by choosing two points and a vector. For example, if $\mathbf{u}_1=(1, 0, 0)$, $\mathbf{u}_2=(0, 0, 1)$, and $\mathbf{u}_3=(0, 1, 1)$, then it can readily be verified that the B-basis functions are

$$b^n_\alpha(x, y, z)=\frac{n!}{\alpha!}x^{\alpha_1}(z-y)^{\alpha_2}y^{\alpha_3}.$$

If a homogeneous polynomial $B(\mathbf{u})$ has coefficients C_α with respect to this hybrid BM basis, that is,

$$B(\mathbf{u}) = \sum_{|\alpha| = n} C_\alpha \frac{n!}{\alpha!} x^{\alpha_1}(z-y)^{\alpha_2} y^{\alpha_3},$$

the coefficient C_{n00} represents, up to constant multiples, the directional derivative of $B(\mathbf{u})$ of order n in the direction of the vector $(1, 0)$. The coefficients $C_{k,\, n-k,\, 0}$ (resp. $C_{l,\, 0,\, n-l}$) represent, up to constant multiples, the directional derivatives of $B(\mathbf{u})$ of order k in the direction of the vector $(1, 0)$ evaluated at the point $(0, 0)$ (resp. the directional derivatives of $B(\mathbf{u})$ of order l in the direction of the vector $(1, 0)$ evaluated at the point $(0, 1)$). This interpretation of the coefficients of a polynomial can be extended easily to the case when the polynomial is expressed in a general hybrid BM basis defined by two points and a vector.

We can also realize a hybrid BM basis as an L-basis. Let $L_1 = a_1 x + b_1 y + c_1 z$, $L_2 = a_2 x + b_2 y + c_2 z$, and $L_3 = a_3 x + b_3 y + c_3 z$ be three linearly independent polynomials. Let us choose the knot-net of polynomials $L_{i,\, j} = L_i$, $1 \leqslant j \leqslant n$. The restriction that $a_1 b_2 - a_2 b_1 \neq 0$, $a_1 b_3 - a_3 b_1 \neq 0$, and $a_2 b_3 - a_3 b_2 \neq 0$ defines a homogeneous Bernstein–Bézier basis. The restriction that $a_3 = b_3 = 0$ defines a multinomial basis. It is easy to verify that the only remaining restriction that maintains linear independence is $a_1 b_2 - a_2 b_1 \neq 0$, $a_2 b_3 - b_2 a_3 \neq 0$, and $a_1 b_3 - a_3 b_1 = 0$. Thus the lines corresponding to L_1 and L_3 are parallel. With this restriction the homogeneous L-basis is referred to as a hybrid BM basis. This hybrid basis is defined by the two points: $(b_1 c_2 - b_2 c_1 / a_1 b_2 - a_2 b_1,\ a_2 c_1 - a_1 c_2 / a_1 b_2 - a_2 b_1)$, $(b_2 c_3 - b_3 c_2 / a_2 b_3 - a_3 b_2,\ a_3 c_2 - a_2 c_3 / a_2 b_3 - a_3 b_2)$ and the vector $(-b_1 / a_1 b_2 - a_2 b_1,\ a_1 / a_1 b_2 - a_2 b_1) = (-b_3 / a_3 b_2 - a_2 b_3,\ a_3 / a_3 b_2 - a_2 b_3) = (b_3 c_1 - b_1 c_3 / \Delta,\ a_1 c_3 - a_3 c_1 / \Delta)$.

4.1.4. *Duality*

This section investigates the duality between bivariate Bernstein–Bézier and multinomial bases. First we describe the algebraic or de Boor–Fix duality between Bernstein–Bézier and multinomial bases. Then we shall comment upon the geometric duality between these bases.

A Bernstein–Bézier B-basis is defined by (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3) with $c_i \neq 0$ for $i = 1, 2, 3$. The dual L-basis is therefore defined by $L_1 = a_1 x + b_1 y + c_1 z$, $L_2 = a_2 x + b_2 y + c_2 z$, and $L_3 = a_3 x + b_3 y + c_3 z$. Depending upon whether zero, one or two of the three terms $a_1 b_2 - a_2 b_1$, $a_2 b_3 - a_3 b_2$, and $a_3 b_1 - a_1 b_3$ are zero, the dual L-basis can be a Bernstein–Bézier basis, a hybrid BM basis, or a multinomial basis. More specifically, if all three terms are non-zero, then the dual L-basis is a Bernstein–Bézier basis; if exactly two of these terms are non-zero, then the dual L-basis is a hybrid BM basis, and finally if exactly one of these three terms is nonzero, then the dual L-basis is a multinomial basis. Note that these distinctions are very sensitive to the choice of the coordinate system. The upper diagram of Fig. 2 presents three Bernstein–Bézier B-bases each defined by three points forming a right-angle triangle. The duals to these Bernstein–Bézier B-bases are shown immediately below them in the lower part of Fig. 2. Depending upon the choice of the coordinate system, the dual bases are a multinomial basis, a BM basis, and a Bernstein–Bézier basis, respectively.

The duality situation is similar for a multinomial B-basis defined by (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3) where exactly two of the three terms c_1, c_2, and c_3 are zero. Again the dual L-basis can be either a Bernstein–Bézier basis, a hybrid BM basis, or a multinomial basis depending upon how many of the three terms $a_1 b_2 - a_2 b_1$, $a_2 b_3 - a_3 b_2$, and $a_3 b_1 - a_1 b_3$ vanish.

In summary, a uniform B-basis—which can be either a Bernstein–Bézier basis, a hybrid BM basis, or a multinomial basis—is dual to a uniform L-basis, which can also be either a Bernstein–Bézier basis, a hybrid BM basis, or a multinomial basis.

These observations lead to the following geometric interpretation of duality between uniform B-bases and uniform L-bases. A Bernstein–Bézier B-basis is defined by three points; a hybrid BM B-basis by two points and a vector; a multinomial B-basis by a point and two vectors. Interpreting a vector as a point at infinity, a uniform B-basis is defined by three points.

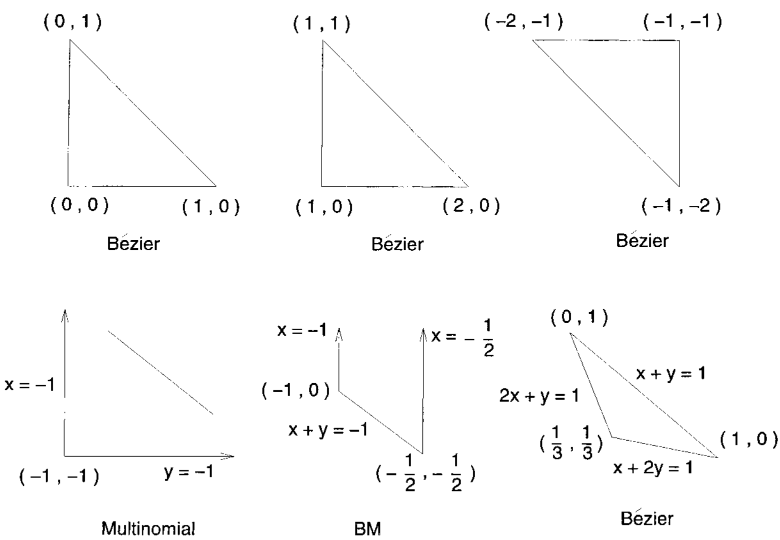

Fig. 2. Duality between Bernstein–Bézier, multinomial, and BM bases.

The dual L-basis is defined by three lines. Notice that the conditions $a_i b_j - a_j b_i = 0$ correspond to parallel lines in affine space and the number of parallel lines leads to the distinction between Bernstein–Bézier, BM, and multinomial L-bases. A Bernstein–Bézier L-basis is defined by three non-parallel lines in the affine plane. A BM L-basis is defined by three lines in the affine plane, exactly two of which are parallel. Finally, a multinomial L-basis is defined by the line at infinity and two non-parallel lines in the affine plane. In projective space where there are no parallel lines, these distinctions disappear.

Observe that it is *not* true that the three cases of uniform L-bases, namely Bernstein–Bézier basis, hybrid BM basis, and multinomial basis, arise by taking i lines in the affine plane and $3-i$ lines at infinity for $i=3, 2, 1$. In fact although there are many points at infinity, there is only one line at infinity. The multinomial L-basis arises by choosing exactly one line at infinity as described above. Alternatively, the three cases of uniform L-bases, namely Bernstein–Bézier basis, hybrid BM basis, and multinomial basis, arise by taking three lines such that i points of intersection of these lines lie in the affine plane and $3-i$ points of intersection lie at infinity for $i=3, 2, 1$, respectively.

There is another potential source of confusion which is intriguing. Observe that the Bernstein–Bézier B-basis defined by the three points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the same as, but *not* dual to, the L-basis defined by the three lines $\mathbf{v}_1\mathbf{v}_2$, $\mathbf{v}_2\mathbf{v}_3$, and $\mathbf{v}_3\mathbf{v}_1$. Such a duality, if it exists, should be referred to as *self-duality*. Under self-duality, the correspondence between vectors in R^3 and the homogeneous polynomials on R^3 would have to be defined from a set of 3-vectors to 3-polynomials and vice versa rather than from a vector to a polynomial. In particular, a triple of vectors (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3) would correspond to the three homogeneous polynomials $(b_2c_3 - b_3c_2)x + (a_3c_2 - a_2c_3)\,y + (a_2b_3 - a_3b_2)z$, $(b_3c_1 - b_1c_3)x + (a_1c_3 - a_3c_1)\,y + (a_3b_1 - a_1b_3)z$, and $(b_1c_2 - b_2c_1)x + (a_2c_1 - a_1c_2)\,y + (a_1b_2 - a_2b_1)z$ under this self-dual correspondence. It would be very interesting to explore this self-duality. However, the duality presented in this work is not self-duality.

4.2. *Duality between Lagrange and Power Bases*

This section establishes that certain proper subclasses of bivariate Lagrange and power bases can be realized respectively as special cases of L-bases and B-bases and then investigates the duality between these special bases.

4.2.1. *Lagrange Bases*

Let $\{\{L_{ij}\}, \{L_{2j}\}, \{L_{3j}\}, j=1, ..., n\}$ be a knot-net of homogeneous polynomials. Suppose that the homogeneous polynomials $(L_{1,\alpha_1+1}, L_{2,\alpha_2+1},$

L_{3,α_3+1}) are linearly dependent for $|\alpha| = n$, $0 \leqslant \alpha_k \leqslant n-1$. The corresponding L-basis is then referred to as a Lagrange L-basis. We shall soon see that these dependency conditions give rise to a point-line configuration with $\binom{n+2}{2}$ points such that each of the $\binom{n+2}{2}$ L-basis functions vanishes at all the points except one, which justifies the terminology Lagrange L-basis.

To observe this, let us analyze the dependency conditions. Overloading the notation, let L_{ij} also denote the lines in the projective plane defined by the equations: $L_{ij} = 0$. The linear dependency condition on the polynomials L_{i,α_i+1} means that the projective lines L_{i,α_i+1} are concurrent for $|\alpha| = n$, $0 \leqslant \alpha_k \leqslant n-1$. Let $\mathbf{v}_\alpha = \bigcap_{k=1}^{3} L_{k,\alpha_k+1}$ for $|\alpha| = n$, $0 \leqslant \alpha_k \leqslant n-1$. These intersections give rise to $\binom{n+2}{2} - 3$ points corresponding to $\binom{n+2}{2} - 3$ dependency conditions. To these points, we shall add three more points: $\mathbf{v}_{n00} = L_{31} \cap L_{21}$, $\mathbf{v}_{0n0} = L_{11} \cap L_{31}$, and $\mathbf{v}_{00n} = L_{11} \cap L_{21}$. It is easy to verify using Eq. (1) that $l^n_\alpha(\mathbf{v}_\beta) = l^n_\alpha(\mathbf{v}_\alpha)\, \delta_{\alpha\beta}$. Therefore, $\{l^n_\alpha / l^n_\alpha(\mathbf{v}_\alpha)\}$ forms a Lagrange basis.

Now we are going to introduce certain interesting point-line configurations that give rise to bivariate Lagrange L-bases. To this extent, let us investigate the dependency conditions more closely in the affine plane. Let P_{ij} be the affine polynomials corresponding to the homogeneous polynomials L_{ij}. Overloading the notation, let P_{ij} also denote the lines in the affine plane defined by the equations: $P_{ij} = 0$. The linear dependency condition on the knot-net of polynomials corresponds to one of the following geometric conditions:

1. The lines (P_{1,α_1+1}, P_{2,α_2+1} and P_{3,α_3+1}) are distinct and concurrent; that is, they all pass through one common point $\mathbf{v}_\alpha = \bigcap_{k=1}^{3} P_{k,\alpha_k+1}$ when $|\alpha| = n$.

2. The lines (P_{1,α_1+1}, P_{2,α_2+1}, and P_{3,α_3+1}) are distinct and parallel. Then $\{L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1}\}$ all pass through a common point at infinity. For example, if $L_{1,\alpha_1+1} = k_1 ax + k_1 by + c_1 z$, $L_{2,\alpha_2+1} = k_2 ax + k_2 by + c_2 z$ and $L_{3,\alpha_3+1} = k_3 ax + k_3 by + c_3 z$, then the common point $\mathbf{v}_\alpha$ is $(-kb, ka, 0)$ for some $k \neq 0$.

3. Only two of the three lines (P_{1,α_1+1}, P_{2,α_2+1} and P_{3,α_3+1}) are distinct. Let $\mathbf{v}_\alpha$ be the point of intersection of the these two lines. If the lines are parallel, then as in case 2, the point of intersection lies at infinity.

4. One of the lines L_{k,α_k+1} lies at infinity. In this case the point of intersection of the lines $\{L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1}\}$ lies at infinity.

Observe that it is not possible to have all three lines the same because this would violate the linear independence condition on the knot-net of polynomials. More specifically, the linear dependence condition for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = n$ and the linear independence condition for $(\alpha_1 - 1, \alpha_2, \alpha_3)$ imply that if the two lines L_{2,α_2+1} and L_{3,α_3+1} are same, then α_1 must be zero. Therefore, if all three lines are the same, then it must be the case that $\alpha_i = 0$ for $i = 1, 2, 3$; that is $n = 0$, in which case there is only one L-basis function. This argument also shows that the condition that two lines are the same is very restrictive and can happen only if one of the three $\alpha_i = 0$. Such cases, however, do arise in practice as we shall see below.

If an affine polynomial $B(\mathbf{v})$ of degree n is represented with respect to an affine Lagrange L-basis, that is,

$$B(\mathbf{v}) = \sum_{|\alpha| = n} C_\alpha P^n_\alpha(\mathbf{v}),$$

the coefficients C_α represent, up to constant multiples, the value of the polynomial $B(\mathbf{v})$ at v_α, whenever v_α is not at infinity. More precisely, $B(\mathbf{v}_\alpha) = C_\alpha P^n_\alpha(\mathbf{v}_\alpha)$. When v_α is at infinity, as in cases 2, 3, and 4 above, it can be verified easily that the coefficients C_α represent, up to constant multiples, the directional derivative of $B(\mathbf{v})$ of order n in the direction of one of the parallel lines that give rise to v_α as the common point of intersection. Observe that since $B(\mathbf{v})$ is a polynomial of degree n, its directional derivative of order n is a constant and, therefore, it does not matter where it is evaluated.

Now we present certain point-line configurations that give rise to Lagrange L-bases. Figure 3 shows a configuration of lines in R^2 for which the dependency conditions are satisfied and all the lines are distinct and concurrent. The configuration of lines in Fig. 3 also satisfies the linear independence condition for $(L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1})$, $0 \leqslant |\alpha| \leqslant n-1$, which is required to define a knot-net of polynomials. Figure 3 is an example of a *principal lattice* or *geometric mesh* [CY77] of order n, which can be described by three sequences of n lines $\{\{L_{1i}\}, \{L_{2j}\}, \{L_{3k}\}, 1 \leqslant i, j, k \leqslant n\}$ such that each set of three lines $\{L_{1,i+1}, L_{2,j+1}, L_{3,k+1}, i+j+k=n\}$

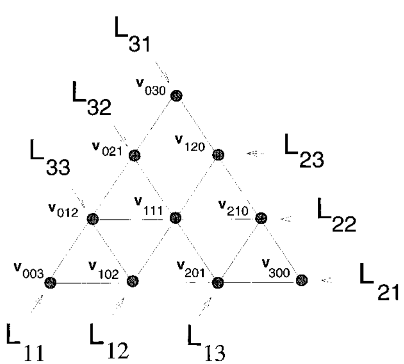

Fig. 3. Geometric mesh of order 3 for Lagrange L-basis.

intersect at exactly one common point $\mathbf{v}_{ijk}$. It is clear from the above construction that every geometric mesh gives rise to a Lagrange L-basis.

Figure 4 shows some configurations of six lines and six points in the projective plane that give rise to a Lagrange L-basis. These are examples of geometric meshes of order 2. The right diagram of Fig. 4 shows a configuration where one of the points is at infinity.

Figure 5 shows some configurations of four lines and six points in the projective plane that give rise to a Lagrange L-basis. In this case, two of the lines in every dependency condition are the same. These are examples of *natural lattices* [CY77] of order n, which are defined by $n+2$ lines in the projective plane such that the $\binom{n+2}{2}$ intersection points of these lines are all distinct. The left, middle, and right diagrams of Fig. 5 show configurations where zero, one, and three points lie at infinity. Since every natural lattice of order n generates a Lagrange basis of degree n, it is natural to ask whether every natural lattice of order n gives rise to a Lagrange *L-basis* of degree n. Unfortunately, the answer is no. Figure 6 shows a natural lattice of order 3. It is easy to verify that it is not possible to realize the Lagrange basis corresponding to this configuration as an L-basis. Thus the Lagrange L-bases form a proper subset of the set of all bivariate Lagrange bases.

4.2.2. *Power Bases*

Let $\{\mathbf{u}_{1j}, \mathbf{u}_{2j}, \mathbf{u}_{3j},\ j=1, ..., n\}$ be a knot-net of vectors. Suppose the vectors $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ are linearly dependent for $|\alpha|=n$, $0 \leqslant \alpha_k \leqslant n-1$. The corresponding B-basis is referred to as a power basis because, as we shall soon see, up to constant multiples every basis function is an nth power of a linear polynomial.

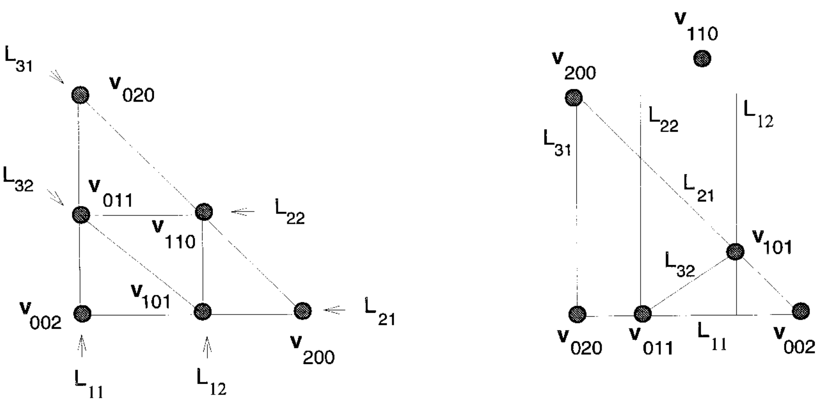

Fig. 4. Geometric mesh of order 2 for Lagrange L-basis.

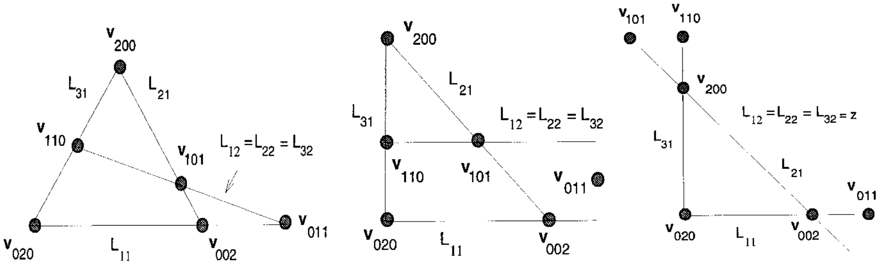

Fig. 5. Natural lattice of order 2 for Lagrange L-basis.

Let $\mathbf{v}_{k,\,\alpha_k+1}$ represent the points in the projective plane corresponding to the vectors $\mathbf{u}_{k,\,\alpha_k+1}$. Then the linear dependency condition on the vectors $\mathbf{u}_{k,\,\alpha_k+1}$ means that the corresponding points $\mathbf{v}_{k,\,\alpha_k+1}$ are collinear in the projective plane. Let Q_α be the line defined by the three collinear points $\mathbf{v}_{k,\,\alpha_k+1}$ for $|\alpha| = n$, $0 \leqslant \alpha_k \leqslant n-1$, and let $q_\alpha = 0$ be the equation of the line Q_α. This construction gives rise to $\binom{n+2}{2} - 3$ lines corresponding to the $\binom{n+2}{2} - 3$ dependency conditions. Now let us add three more lines. Define Q_{n00}, Q_{0n0}, and Q_{00n} to be the lines passing through the points $\mathbf{v}_{21}\mathbf{v}_{31}$, $\mathbf{v}_{11}\mathbf{v}_{31}$, and $\mathbf{v}_{11}\mathbf{v}_{21}$ respectively, and let q_{n00}, q_{0n0}, and q_{00n} be the equations of these lines. In the Appendix we give an inductive proof that the B-basis functions b^n_α for the knot-net $\mathbf{u}_{ij}$ are equal to $(q_\alpha)^n$ up to constant multiples. In the next section, we shall give a much simpler proof of this fact based on the duality between the Lagrange and power bases.

Figure 7 shows two configurations of points in R^2 for which the dependency conditions are satisfied because the points $\mathbf{v}_{k,\,\alpha_k+1}$ for $|\alpha| = n$ are collinear. The configurations of points in Fig. 7 also satisfy the linear independence condition for $\mathbf{u}_{k,\,\alpha_k+1}$, $k = 1, 2, 3$, $|\alpha| \leqslant n-1$, which is required to define a knot-net of vectors. This figure is an example of a *dual principal lattice* or *dual geometric mesh* of order n, which is defined by $3n$ distinct points $\{\mathbf{v}_{1j}, \mathbf{v}_{2j}, \mathbf{v}_{3j}, j = 1, \ldots, n\}$ such that each set of three points $\{\mathbf{v}_{1,\,i+1}, \mathbf{v}_{2,\,j+1}, \mathbf{v}_{3,\,k+1}, i+j+k = n\}$ is collinear and defines the line Q_{ijk}. The seven lines defined by the dependency conditions are shown as dark lines while the remaining three add-on lines are shown as dotted lines. It is clear from the above construction that every dual geometric mesh gives rise to a power B-basis.

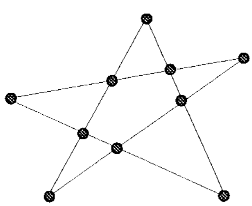
Fig. 6. Natural lattice of order 3 that does not admit an L-basis.

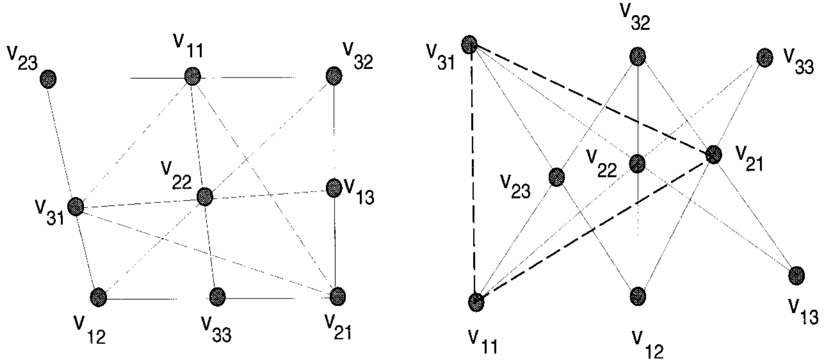

Fig. 7. Dual geometric mesh of order 3 for power *B*-basis.

Figure 8 shows examples of point-line configurations with six points and six lines that give rise to power *B*-bases. These are examples of dual geometric meshes of order 2. The right diagram of Fig. 8 shows a configuration where one of the points lies at infinity and this is represented by a vector in the affine plane.

Figure 9 shows some configurations of six lines and four points in the projective plane that give rise to a power *B*-basis. These are examples of *dual natural lattices* of order 2. A *dual natural lattice* of order n is defined by $n+2$ distinct points and $\binom{n+2}{2}$ distinct lines joining these points. The

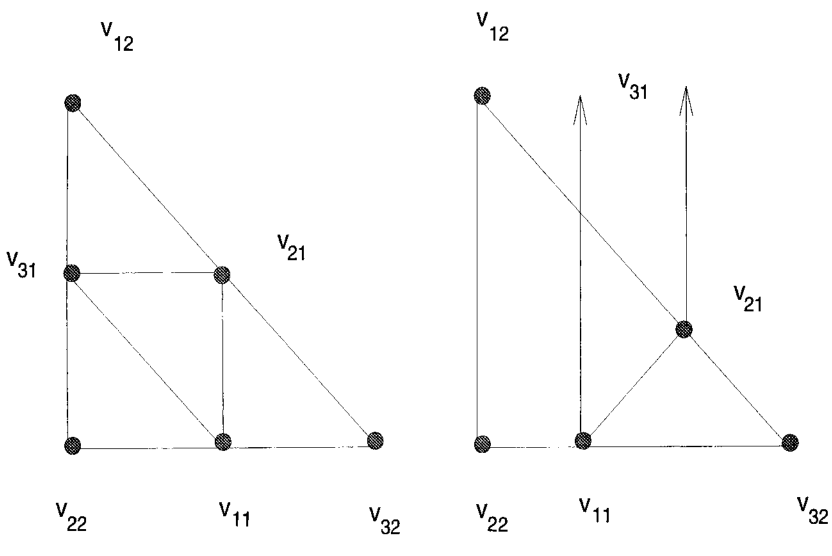

Fig. 8. Dual geometric mesh of order 2 for power *B*-basis.

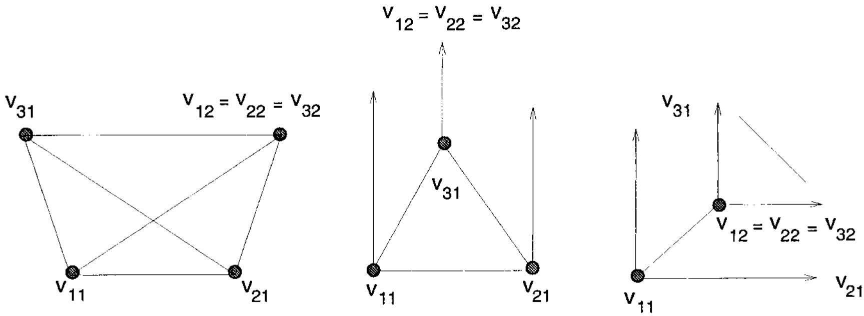

Fig. 9. Dual natural lattice of order 2 for power B-basis.

left, middle, and right diagrams of Fig. 9 show configurations of points and lines, where zero, one, and two points lie at infinity and these are represented by vectors in the affine plane. Since every dual natural lattice of order n generates a power basis of degree n, it is natural to ask whether every dual natural lattice of order n gives rise to a power B-basis of degree n. Unfortunately, the answer is no. It is easy to verify by exhaustive enumeration that the configuration of points and lines corresponding to the dual natural lattice of order 3 shown in Fig. 10 cannot be realized as a B-basis. A simpler proof based on duality will be given at the end of Section 4.2.3. Thus the power B-bases form a proper subset of the set of all bivariate power bases.

4.2.3. *Duality*

Let a Lagrange L-basis be defined by a knot-net $\mathscr{L}$ of polynomials $\{L_{ij}, L_{2j}, L_{3j}, j=1, ..., n\}$ as in Section 4.2.1, and let the $\binom{n+2}{2}$ nodes corresponding to this Lagrange L-basis be denoted by $\mathbf{v}_\alpha$. Let the dual

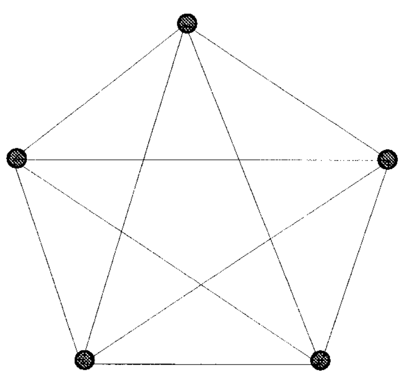

Fig. 10. Dual natural lattice of order 3.

B-basis be defined by the knot-net $\mathscr{U}$ of vectors $\{\mathbf{u}_{ij}, \mathbf{u}_{2j}, \mathbf{u}_{3j}, j=1, ..., n\}$ under the knot net correspondence $(a, b, c) \leftrightarrow ax+by+cz$ defined in Section 3.1 so that $L_{ij}(\mathbf{u})=\mathbf{u}\cdot\mathbf{u}_{ij}$. It is clear that both the linear independence conditions and the linear dependence conditions are preserved under this correspondence. In particular, the linear dependency condition or the collinearity condition on a set of three points used for defining B-bases corresponds to the linear dependency condition or the concurrency condition on the corresponding set of three lines used for defining L-bases. Therefore, the dual B-basis is a power basis as defined in Section 4.2.2.

The Cavaretta–Micchelli identity mentioned in Section 3.1 provides a very simple proof that the B-basis dual to a Lagrange L-basis is a power basis, that is, that every element of the B-basis is an nth power of a linear polynomial. Indeed given a Lagrange L-basis, it was verified in Section 4.2.1 that the L-basis functions $\{l^n_\alpha\}$ satisfy the relation $l^n_\alpha(\mathbf{v}_\beta)=l^n_\alpha(\mathbf{v}_\alpha)\,\delta_{\alpha\beta}$. Substituting this identity into the Cavaretta–Micchelli identity, we obtain $b^n_\alpha(\mathbf{u})=(\mathbf{v}_\alpha\cdot\mathbf{u})^n/l^n_\alpha(\mathbf{v}_\alpha)$, which establishes that up to constant multiples each element of the dual B-basis is an nth power of a linear polynomial. Using the definition of the L-basis functions given in Eq. (1) together with the fact that by duality $q_\alpha=\mathbf{v}_\alpha\cdot\mathbf{u}$, we can also rewrite the power B-basis functions as

$$b^n_\alpha(\mathbf{u})=\frac{(q_\alpha)^n}{\prod_{j=1,\ldots,\alpha_i;\, i=1,2,3} q_\alpha(\mathbf{u}_{ij})}.$$

Notice that the Cavaretta–Micchelli identity and the generalized de Boor–Fix formula hold for all bivariate Lagrange and power bases, even though these bases may not be L-bases and B-bases, respectively. The argument in the preceding paragraph can be used to establish this general duality between bivariate Lagrange and power bases.

To appreciate the geometry of this correspondence, notice that a Lagrange L-basis of degree n is defined, in general, by $3n$ lines while a power B-basis of degree n is defined, in general, by $3n$ points. However, these lines (in case of the Lagrange L-basis) and points (in case of the power B-basis) need not be distinct. Such is the case, for example, with the natural lattice and dual natural lattice configurations, where certain lines in case of the Lagrange L-basis and certain points in the case of power B-basis do coincide.

The geometric mesh configuration of order 3 for a Lagrange L-basis depicted in Fig. 3 consisting of 9 distinct lines and 10 distinct points is dual to the dual geometric mesh configuration of order 3 for a power B-basis depicted in Fig. 7 consisting of 9 distinct points and 10 distinct lines.

Similarly, the geometric mesh configuration of order 2 for a Lagrange L-basis depicted in Fig. 4 consisting of six distinct lines and six distinct points is dual to the dual geometric mesh configuration of order 2 for a power B-basis depicted in Fig. 8 consisting of six distinct points and six distinct lines. The distinction between different cases as to whether certain points lie at infinity or whether certain lines are parallel disappears in projective space. Figures 5 and 9 are dual to themselves. In such a configuration, the number of lines must be equal to the number of points. Since $3n = \binom{n+2}{2}$ only for $n = 1, 2$, these are the only situations where the geometric mesh configuration is self-dual. Figure 7 shows the dual geometric mesh for $n = 3$ which is not self-dual.

The natural lattice configuration of order 2 depicted in Fig. 4 consisting of four lines and six points for the Lagrange L-basis is dual to the dual natural lattice configuration of six lines and four points for the power B-basis depicted in Fig. 8. Finally, the natural lattice configuration depicted in Fig. 6 consisting of 5 lines and 10 points cannot be realized as a Lagrange L-basis. Therefore by duality the dual natural lattice configuration of 10 lines and 5 points depicted in Fig. 10 cannot be realized as a power B-basis.

4.3. *Duality between Newton and Newton Dual Bases*

This section establishes that certain subclasses of bivariate Newton bases can be realized as special cases of L-bases. We then introduce the Newton dual bases and investigate the duality between the Newton and Newton dual bases.

4.3.1. *Newton Bases*

Suppose the following restriction is imposed on the knot-net $\mathscr{L}$ of homogeneous polynomials: $L_{1i} = a_{1i}x + b_{1i}y + c_{1i}z$, $L_{2i} = a_{2i}x + b_{2i}y + c_{2i}z$, and $L_{3i} = a_3 x + b_3 y + c_3 z$. In other words, one of the three sequences of knots consists of one and the same polynomial. The corresponding L-basis will be referred to as the generalized bivariate homogeneous Newton L-basis. We shall be interested here in the special case where $L_{3i} = z$; the corresponding L-basis will be referred to as the bivariate homogeneous Newton L-basis. Observe that the multinomial basis is a special case of the Newton basis. By choosing the polynomials $L_{1i} = x - a_i z$, $L_{2i} = y - b_j z$, $L_{3i} = z$, and dehomogenizing, we obtain the following corresponding affine Newton basis:

$$l_\alpha^n = \prod_{i=1}^{\alpha_1} (x - a_i) \prod_{j=1}^{\alpha_2} (y - b_j).$$

For example when $n = 2$ this construction yields the basis functions: 1, $(x - a_1)$, $(x - a_1)(x - a_2)$, $(y - b_1)$, $(y - b_1)(y - b_2)$, and $(x - a_1)(y - b_1)$.

Therefore, the affine Newton basis for surfaces defined here is a generalization of the affine Newton basis for curves.

To justify the terminology Newton basis, we are going to establish that these Newton L-bases are special cases of the bivariate Newton bases defined by Gasca [Gas90]. Gasca starts with a particular set of lines and points and associates a Newton basis to this point-line configuration. In contrast, our construction proceeds in the opposite direction.

We have other stronger incentives for establishing this connection. We plan to construct certain point-line configurations and associated point and derivative interpolation problems, which give rise to Newton L-bases in a natural way. To be more precise: to each Newton L-basis, we wish to associate an interpolation problem for point and derivative data with the following properties: (i) there exists a unique solution to the general interpolation problem and (ii) the coefficients a_α of the interpolant $L(\mathbf{u}) = \sum_{|\alpha|=n} a_\alpha l_\alpha^n$ expressed in the Newton L-basis are the solutions of a lower triangular system of linear equations. This task is complicated, however, by the fact that given a suitable point-line configuration, the associated point and derivative interpolation problem is not unique. This non-uniqueness is intrinsic to the bivariate Newton basis and is true as well for the univariate Newton basis.

To gain some insight into this important point, we explain the nature and cause of non-uniqueness in the case of curves by presenting a simple example. To the univariate affine Newton basis of degree 2 given by 1, $(x-a_1)$, and $(x-a_1)(x-a_2)$, $a_1 \neq a_2$, one can associate a point interpolation problem at a_1 and a_2, but for the third interpolation condition one can choose any arbitrary point a_3 or in fact even the derivative at a_2. Indeed if $f(x) = c_0 + c_1(x-a_1) + c_2(x-a_1)(x-a_2)$, then $c_0 = f(a_1)$, and $c_1 = f[a_2, a_1]$, where

$$f[a_2, a_1] = \frac{f(a_2) - f(a_1)}{(a_2 - a_1)}$$

is the usual divided difference. More interestingly, $c_2 = f[a_3, a_2, a_1]$ where

$$f[a_3, a_2, a_1] = \frac{f[a_3, a_2] - f[a_2, a_1]}{(a_3 - a_1)} \qquad \text{if} \quad a_3 \neq a_2,$$

$$f[a_2, a_2, a_1] = \frac{f'(a_2) - f[a_2, a_1]}{(a_2 - a_1)} \qquad \text{if} \quad a_3 = a_2,$$

where $f'(a_2)$ denotes the first derivative of f at a_2. Similarly, if the Newton basis is $1, (x-a_1)$, and $(x-a_1)^2$, then $c_0 = f(a_1), c_2 = f'(a_1)$, and $c_2 = f[a_2, a_1, a_1]$, where a_2 is any arbitrary point including a_1.

This freedom in choosing the interpolation problem carries over to the bivariate setting. Although Gasca [Gas90] observes that there is some freedom, his construction does not clarify the role of freedom in choosing the lines and points. For our purposes, it is essential to explore the nature of this non-uniqueness in order to specify certain interesting point-line configurations associated with Newton L-bases.

To associate an interpolation problem with a Newton L-basis, we first introduce a set of points. To this purpose, observe that the linear independence conditions on the knot-net of polynomials imply that the lines L_{1,α_1+1} and L_{2,α_2+1} are distinct and non-parallel for $0 \leqslant \alpha_1+\alpha_2 \leqslant n-1$. Let $\mathbf{v}_{\alpha_1,\alpha_2,\alpha_3} = L_{1,\alpha_1+1} \cap L_{2,\alpha_2+1}$ for $0 \leqslant \alpha_1+\alpha_2 \leqslant n-1$. These points could be distinct or the same depending upon the lines themselves, but, in any event we get $\frac{1}{2}n(n+1)$ points counted with appropriate multiplicity. Next we introduce an additional $n+1$ points for a total of $\binom{n+2}{2}$ points again counted with appropriate multiplicity. The choice of the remaining $n+1$ points $\mathbf{v}_{\alpha_1,\alpha_2,0}$ is more subtle and incorporates the freedom of choice discussed in the previous paragraph. One way to select these additional $n+1$ points is to choose the point $\mathbf{v}_{\alpha_1,\alpha_2,0}$ to be any point on the line L_{2,α_2+1} for $0 \leqslant \alpha_2 \leqslant n-1$ and let $\mathbf{v}_{0,n,0}$ to be any arbitrary point. This freedom in choosing the $(n+1)$ points can also be described by selecting additional lines $L_{2,n+1}$ and $G_1, ..., G_{n+1}$, $G_i \neq L_{2,i}$ so that $\mathbf{v}_{\alpha_1,\alpha_2,0} = L_{2,\alpha_2+1} \cap G_{\alpha_2+1}$ for $0 \leqslant \alpha_2 \leqslant n-1$, and $\mathbf{v}_{0,n,0} = L_{2,n+1} \cap G_{n+1}$. Alternatively, one can select these additional $n+1$ points $\mathbf{v}_{\alpha_1,\alpha_2,0}$ to be any point on the line L_{1,α_1+1} for $0 \leqslant \alpha_1 \leqslant n-1$ and let $\mathbf{v}_{n,0,0}$ be any arbitrary point. This freedom in choosing the $(n+1)$ points can also be described by selecting additional lines $L_{1,n+1}$ and $F_1, ..., F_{n+1}$, $F_i \neq L_{1,i}$ so that $\mathbf{v}_{\alpha_1,\alpha_2,0} = L_{1,\alpha_1+1} \cap F_{\alpha_1+1}$ for $0 \leqslant \alpha_1 \leqslant n-1$, and $\mathbf{v}_{n,0,0} = L_{1,n+1} \cap F_{n+1}$. For sake of definiteness, and without loss of generality, we shall assume that we have opted for this latter choice while associating an interpolation problem in the next paragraph. Observe that combining the two alternatives, one obtains a simplified symmetric choice by picking $\mathbf{v}_{n00}$ and $\mathbf{v}_{0n0}$ to be arbitrary points on the lines L_{21} and L_{11}, respectively, and choosing (for the remaining points) $\mathbf{v}_{\alpha_1,\alpha_2,0} = L_{1,\alpha_1+1} \cap L_{2,\alpha_2+1}$ whenever L_{1,α_1+1} and L_{2,α_2+1} are distinct; otherwise, if they are the same line, choose any point on this line. Thus we can associate a total of $\binom{n+2}{2}$ points counted with appropriate multiplicity to a Newton L-basis defined by $2n$ lines. This point-line configuration associated with a Newton L-basis is a subclass of the point-line configurations that form the starting point for the construction of Newton bases defined by Gasca [Gas90]. With this associated point-line configuration, it can be readily verified that the Newton L-bases defined here can be realized as special cases of the bivariate Newton bases defined by Gasca.

Now we are in a position to describe the interpolation system associated with this Newton L-basis. Our procedure is exactly the same as in

[Gas90]. Let s_α be the number of functions in the set $\{L_{1,1}, ..., L_{1,\alpha_1}, L_{2,1}, ..., L_{2,\alpha_2}\}$ that vanish at v_α and coincide with L_{1,α_1+1} up to constant factors. Let t_α be the number of functions in the set $\{L_{1,1}, ..., L_{1,\alpha_1}, L_{2,1}, ..., L_{2,\alpha_2}\}$ that vanish at v_α and *do not* coincide with L_{1,α_1+1} up to constant factors. The associated interpolation problem is to interpolate the following point and derivative information:

$$\frac{\partial^{s_\alpha+t_\alpha} f(\mathbf{v}_\alpha)}{\partial^{s_\alpha} L_{1,\alpha_1+1}\, \partial^{t_\alpha} L_{2,\alpha_2+1}} \qquad \text{for} \quad 0 \leqslant \alpha_1 + \alpha_2 \leqslant n-1,$$

$$\frac{\partial^{s_\alpha+t_\alpha} f(\mathbf{v}_\alpha)}{\partial^{s_\alpha} L_{1,\alpha_1+1}\, \partial^{t_\alpha} F_{\alpha_2+1}} \qquad \text{for} \quad 0 \leqslant \alpha_1 + \alpha_2 = n,$$

where $\partial f/\partial L = b(\partial f/\partial x) - a(\partial f/\partial y)$, when $L = ax + by + c$.

It is not too difficult to prove that this interpolation problem has a unique solution and that the interpolant expressed in terms of the Newton L-basis can be found by solving a lower triangular system of linear equations. The proof of this fact is also described by Gasca [GM89] and is therefore omitted here. In fact, the coefficients of the solution can be interpreted as the generalization of divided differences to higher dimensions. Further discussion of the extremely important role the Newton bases play in multivariate interpolation and approximation can be found in [Gas90].

The configurations of lines and points corresponding to Newton L-bases are very flexible. By choosing two sequences of parallel lines $L_{1,i}$ and $L_{2,j}$ as shown in Fig. 11, selecting the symmetric choice of the associated interpolation problem, and picking the points $\mathbf{v}_{300}$ and $\mathbf{v}_{030}$ as indicated in Fig. 11, it is clear that every geometric mesh gives rise to a Newton L-basis. Recall from Section 4.2 (Fig. 3) that the same configuration also gives rise to a Lagrange L-basis.

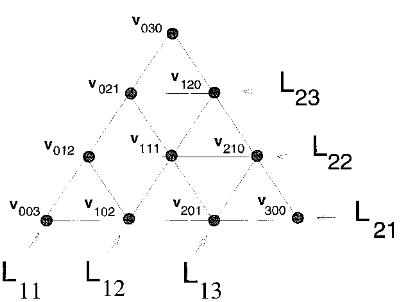

Fig. 11. Point interpolation using Newton L-basis.

More interestingly, every natural lattice of order n also gives rise to a Newton L-basis. Figure 6 shows a natural lattice of order 3. This lattice gives rise to the Newton L-basis by choosing $L_{1,i} = L_i$ for $1 \leqslant i \leqslant n$, $L_{2,j} = L_{n+3-j}$, $1 \leqslant j \leqslant n$, selecting the symmetric choice, and picking the point $\mathbf{v}_{0n0}$ on L_1 as $L_1 \cap L_2$ and the point $\mathbf{v}_{n00}$ on L_{n+2} as $L_{n+2} \cap L_{n+1}$.

It is not yet clear whether or not every configuration of lines and points that gives rise to a Lagrange L-basis also gives rise to a Newton L-basis. More interestingly, one can ask whether a configuration of lines and points satisfying the GC condition [CY77], HGC condition [Bus85], or DH conditions [CL88] always gives rise to a Newton L-basis. The HGC conditions, in particular, generalize the Hermite interpolation conditions to higher dimensions and the DH conditions generalize the GC conditions still further.

Newton's representation of the solution of the Hermite interpolation problem has also been considered by several researchers in the past [GM87, Jet83, GM89, Mae82, GR84]. A popular Hermite problem corresponding to point interpolation at four points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and the two first-order partial derivatives at the three points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is depicted in Fig. 12. Even for this simple Hermite case, it is non-trivial to demonstrate that it can be realized as a Newton L-basis. The choice $L_{11} = L_{13}$, $L_{21} = L_{22}$ as shown in Fig. 12 yields the Hermite interpolation problem by picking the following lines $F_1 = L_{23}$, $F_2 = M$, $F_3 = L_{23}$, $F_4 = L_{21}$, and $L_{14} = L_{12}$, and considering the intersection points $F_i \cap L_{1i}$ for $i = 1, 2, 3, 4$. This is a non-symmetric choice. By enumerating all the possibilities, one can verify that it is not possible to generate a Newton L-basis corresponding to this interpolation problem with any symmetric choice. The associated interpolation data to this Newton L-basis is $\{f(\mathbf{v}_1), \partial f(\mathbf{v}_1)/\partial L_{21}, f(\mathbf{v}_2), \partial f(\mathbf{v}_2)/\partial L_{23}, f(\mathbf{v}_3), \partial f(\mathbf{v}_3)/\partial L_{21}, f(\mathbf{v}_4), \partial f(\mathbf{v}_1)/\partial L_{11}, \partial f(\mathbf{v}_3)/\partial L_{11}, \partial f(\mathbf{v}_2)/\partial L_{12}\}$, which is equivalent to the interpolation data associated with Hermite problem: $\{f(\mathbf{v}_1), f(\mathbf{v}_2), f(\mathbf{v}_3), f(\mathbf{v}_4), \partial f(\mathbf{v}_1)/\partial x, \partial f(\mathbf{v}_1)/\partial y, \partial f(\mathbf{v}_2)/\partial x, \partial f(\mathbf{v}_2)/\partial y, \partial f(\mathbf{v}_3)/\partial x, \partial f(\mathbf{v}_3)/\partial y\}$.

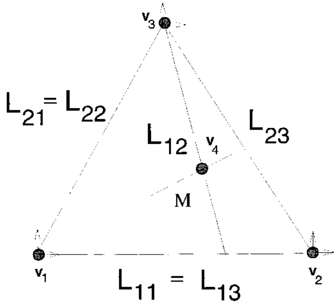

Fig. 12. Hermite interpolation using Newton L-basis.

4.3.2. *Newton Dual Bases*

The homogeneous generalized Newton dual basis is defined as the B-basis functions obtained by imposing the following restrictions on the knot-net $\mathscr{U}$ of vectors: $\mathbf{u}_{1i}=(a_{1i}, b_{1i}, c_{1i})$, $\mathbf{u}_{2i}=(a_{2i}, b_{2i}, c_{2i})$, and $\mathbf{u}_{3i}=(a_3, b_3, c_3)$. In other words, the generalized Newton dual basis is obtained by restricting one of the three sequences of vectors in the knot-net to contain exactly one element. The homogeneous Newton dual basis for surfaces defined here is a generalization of the Newton dual basis for curves [BG93]. Observe that the Bernstein–Bézier, multinomial, and BM B-bases are special cases of the Newton dual basis. Another important subclass of the generalized Newton dual basis is obtained by imposing the following restrictions on the knot-net: $\mathbf{u}_{1i}=(a_{1i}, b_{2i}, 1)$, $\mathbf{u}_{2i}=(1, 0, 0)$, and $\mathbf{u}_{3i}=(0, 1, 0)$. One interesting and useful property of this Newton dual basis is that in the up recurrence relation for the B-basis defined in Section 2.2 by Eq. (2), the labels $h_{k,\alpha}(\mathbf{u})$ do not involve any divisions. Indeed, if $\mathbf{u}=(x, y, z)$, then $h_{1,\alpha}=z$, $h_{2,\alpha}(\mathbf{u})=x-a_{1,\alpha_1+1}z$, $h_{3,\alpha}(\mathbf{u})=y-b_{1,\alpha_1+1}z$. This property of the Newton dual basis can be applied to minimize divisions and simplify computations in change of basis algorithms.

For the sake of completeness, we describe an explicit expression for these special Newton dual basis functions. Let $L_i=y-b_{1i}z$ and $M_i=x-a_{1i}z$. Then the dual basis functions b_α^n are given by

$$b_\alpha^n=\sum L_1^{\alpha_{21}}\cdots L_{\alpha_1+1}^{\alpha_2,\alpha_1+1} M_1^{\alpha_{31}}\cdots M_{\alpha_1+1}^{\alpha_3,\alpha_1+1} z^{\alpha_1},$$

where the sum is taken over all $\alpha=(\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha|=n$ and $\alpha_2=\alpha_{21}+\cdots+\alpha_{2,\alpha_1+1}$, $\alpha_3=\alpha_{31}+\cdots+\alpha_{3,\alpha_1+1}$, and $\alpha_{ij}\geqslant 0$. The derivation is straightforward from the definition, although the bookkeeping is somewhat tedious.

4.3.3. *Duality*

Under the knot-net correspondence it is clear from the construction that the generalized Newton basis is dual to the generalized Newton dual basis. However it is *not* true that the Newton basis is dual to the Newton dual basis. Nevertheless, it is these special cases of the Newton basis and Newton dual basis that turn out to be the most useful in practical situations and hence the terminology. The duality here is similar to the duality we encountered for uniform bases, where a uniform L-basis is dual to a uniform B-basis, although a Bernstein–Bézier basis could be dual to either a Bernstein–Bézier, a multinomial, or a hybrid BM basis. Similarly, the generalized Newton basis is dual to a generalized Newton dual basis, although a Newton basis itself may not necessarily be dual to a Newton dual basis.

5. APPLICATIONS OF DUALITY

There are many applications of duality between B-bases and L-bases. We can use geometric duality to show that a particular point-line configuration can (cannot) represent the knot-net for a B-basis (L-basis) by showing that the dual configuration can (cannot) represent the knot-net for the dual L-basis (B-basis). We used this argument in Section 4.2.3 to conclude that the dual natural lattice of order 3 (Fig. 10) cannot represent the knot-net of a power B-basis because we already knew that the natural lattice of order 3 (Fig. 6) does not represent the knot-net of any Lagrange L-basis.

We can also use algebraic or de Boor–Fix duality to great effect. By applying algebraic duality, we can show that many formulas and algorithms for B-bases map to dual formulas and algorithms for L-bases and vice versa. Thus once we develop a formula or algorithm for one type of basis we can often obtain, almost for free, a dual formula or algorithm for the dual basis. Formulas and algorithms for change of bases [LG95a], evaluation [LG95c], differentiation [LG94a], degree elevation [LG94a], and subdivision [LG95b] each have dual analogues for B-bases and L-bases., This observation allows us to develop a formula or algorithm for whichever scheme is easier to analyze and then map this to a dual formula or algorithm for the dual scheme.

A general change of basis algorithm for B-bases is easy to derive via blossoming [LG95a]. By de Boor–Fix duality, we can use this procedure to construct a dual change of basis algorithm for L-bases [LG95a]. As an application of the constructions in this paper, in the next section we apply this algorithm to convert a bivariate polynomial from a Lagrange representation to a Bernstein–Bézier representation. We also observe that the inverse transformation from Bernstein–Bézier to Lagrange form yields a fast evaluation algorithm for Bernstein–Bézier patches and hence as well for arbitrary B-patches and L-patches.

5.1. *A Change of Basis Algorithm for L-bases*

The computational complexity of general change of basis algorithms from one bivariate polynomial basis of degree n to another bivariate polynomial basis of degree n using matrix multiplication is, in general, $O(n^4)$. Using blossoming and duality, we have derived change of basis algorithms with computational complexity $O(n^3)$ between any two B-bases, any two L-bases, and between any B-basis and any L-basis [LG95a]. These change of basis algorithms are extensions of the change of basis algorithms between any two univariate progressive bases, any two univariate Pólya bases, and between any univariate progressive basis and any univariate Pólya basis [GB92, BG91]. In this work we have demonstrated that certain bivariate Lagrange bases and Newton bases can be realized as L-bases and that certain power bases and Newton dual bases can be realized as B-bases. As a consequence, these change of basis algorithms can now be applied to convert between Bernstein–Bézier, multinomial, Lagrange, power, Newton, and Newton dual bases.

We shall now describe a specific example of a change of basis algorithm from a Lagrange L-basis to a Bernstein–Bézier L-basis to illustrate the general procedure.

Suppose we are given the coefficients R_α of a quadratic polynomial L with respect to the L-basis $\{l_\alpha^n\}$ defined by the knot-net $\mathscr{L} = \{\{L_{1j}\}, \{L_{2j}\}, \{L_{3j}\}, j = 1, 2\}$, where

$$\begin{aligned} L_{11} &= x; & L_{12} &= x - \tfrac{1}{2}; \\ L_{21} &= y; & L_{22} &= y - \tfrac{1}{2}; \\ L_{31} &= 1 - x - y; & L_{32} &= \tfrac{1}{2} - x - y. \end{aligned}$$

The Lagrange L-basis is then given by $l_{200}^2 = x(x - \frac{1}{2})$, $l_{020}^2 = y(y - \frac{1}{2})$, $l_{002}^2 = (1 - x - y)(\frac{1}{2} - x - y)$, $l_{110}^2 = xy$, $l_{101}^2 = x(1 - x - y)$, and $l_{011}^2 = y(1 - x - y)$. The point-line configuration associated with this Lagrange L-basis is shown in the left diagram of Fig. 4.

We would like to compute the coefficients U_α of this polynomial L with respect to the Bernstein–Bézier L-basis $\{p_\alpha^n\}$ defined by another knot-net $\mathscr{M} = \{\{M_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, 2\}$, where

$$\begin{aligned} M_{11} &= x; & M_{12} &= x; \\ M_{21} &= y; & M_{22} &= y; \\ M_{31} &= 1 - x - y; & M_{32} &= 1 - x - y. \end{aligned}$$

The Bernstein–Bézier L-basis is then given by $P_{200}^2 = x^2$, $p_{020}^2 = y^2$, $p_{002}^2 = (1 - x - y)^2$, $p_{110}^2 = xy$, $p_{101}^2 = x(1 - x - y)$, $p_{011}^2 = y(1 - x - y)$.

To describe the change of basis algorithm, we will construct three tetrahedra. We first explain the labeling scheme for these tetrahedra. For each tetrahedron, $(3 - i)(4 - i)/2$ nodes are placed at the ith level of the tetrahedron for $i = 0, 1, 2$ and the nodes along one of the lateral faces are indexed by α for $|\alpha| = 2$. An arrow is placed pointing downward from a node α at the ith level to the three nodes $\alpha + e_1 - e_3$, $\alpha + e_2 - e_3$, and $\alpha - e_3$ at the $(i - 1)$th level directly below it. This labeling scheme for the nodes is shown in Fig. 13. Values, referred to as *labels*, are placed along the arrows. The labels are indexed as $g_{k,\alpha}$ for $k = 1, 2, 3$ and $|\alpha| = 0, 1, 2$ for an arrow from a node $(\alpha_1, \alpha_2, 2 - |\alpha|)$ at the $|\alpha|$th level to the three nodes below it. This labeling scheme for the labels and the arrows is also shown in Fig. 13.

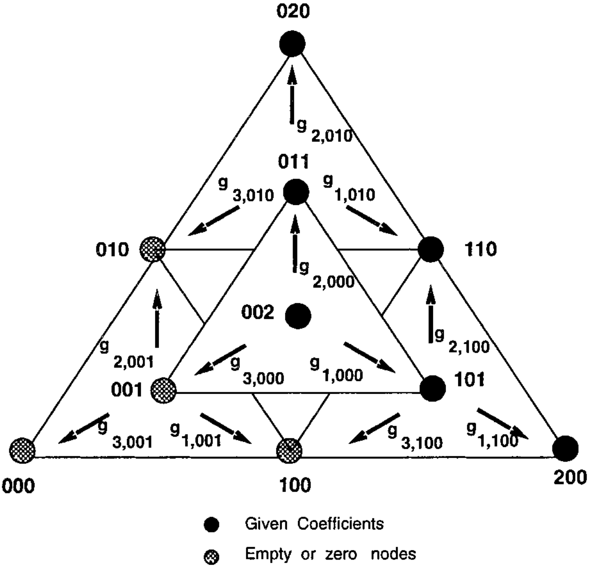

Fig. 13. Labeling of the tetrahedron.

For the first tetrahedron the known coefficients R_α with $|\alpha| = 2$ are placed at the nodes along one of the lateral faces of the tetrahedron as depicted in the first diagram of Fig. 14. The labels $g_{k,\alpha}$ are computed as follows: for $|\alpha| = 0, 1$, let $i = 2 - |\alpha|$; then

$$L_{3i} = g_{1,\alpha} L_{1,\alpha_1+1} + g_{2,\alpha} L_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}.$$

Thus finding $g_{k,\alpha}$ amounts to solving a 3×3 system of linear equations. For our example, the labels are: $g_{1,100} = g_{2,100} = -\frac{1}{2}$, $g_{3,100} = \frac{1}{2}$, $g_{3,100} = g_{3,010} = g_{3,001} = 1$. The rest of the labels are zero. These labels are shown in the first diagram of Fig 14. The computation is now carried out as follows. At the start all the nodes at all levels of the pyramid are empty or

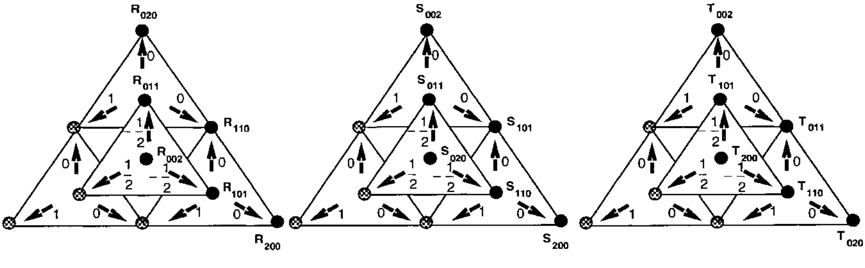

Fig. 14. Change of basis from Lagrange L-basis to Bernstein–Bézier L-basis.

zero other than the nodes α with $|\alpha| = 2$, where the coefficients R_α are placed. The empty or zero nodes are shown as hatched circles in Figs. 13 and 14. The computation starts at the apex of the tetrahedron and proceeds downwards. A value at any empty node is computed by multiplying the label along each arrow that enters the node by the value of the node from which the arrow emerges and adding the results. A value at any non-empty node is computed by applying the same procedure and simply adding the value already at that node. After the computation is complete, the new coefficients $S_{\alpha+(2-|\alpha|)e_3}$ emerge on the nodes α at the base triangle. These coefficients are as follows: $S_{200} = R_{200}$, $S_{110} = R_{110}$, $S_{020} = R_{020}$, $S_{101} = -\frac{1}{2}R_{002} + R_{101}$, $S_{011} = -\frac{1}{2}R_{002} + R_{011}$, $S_{002} = \frac{1}{2}R_{002}$. These new coefficients now express the polynomial L with respect to the L-basis defined by the knot-net $\{\{L_{1j}\}, \{L_{2j}\}, \{M_{3j}\}, j = 1, 2\}$.

We now repeat the above procedure with a second tetrahedron, where the coefficients S_α are placed at the nodes α with $|\alpha| = 2$ as shown in the middle diagram of Fig. 14. The labels on the tetrahedron are permuted from (i, j, k) to (i, k, j) because we now wish to retain the polynomial M_{3j} and replace the polynomials L_{2j} by M_{2j}. The labels $g_{k,\alpha}$ are now computed as follows: For $|\alpha| = 0, 1$, let $i = 2 - |\alpha|$; then

$$L_{2i} = g_{1,\alpha} L_{1,\alpha_1+1} + g_{2,\alpha} M_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}.$$

These labels are also shown in the middle diagram of Fig. 14 and in our special case turn out to be the same as in the first tetrahedron. After the computation is complete, the new coefficients T_α emerge on the nodes at the base triangle. These coefficients are as follows: $T_{200} = S_{200}$, $T_{110} = -\frac{1}{2}S_{020} + S_{110}$, $T_{020} = \frac{1}{2}S_{020}$, $T_{101} = S_{101}$, $T_{011} = -\frac{1}{2}S_{020} + S_{011}$, $T_{002} = S_{002}$. These coefficients now express the polynomial L with respect to the L-basis defined by the knot-net $\{\{L_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, 2\}$

Finally we repeat the above procedure with a third tetrahedron, where the coefficients T_α are now placed at the nodes α with $|\alpha| = 2$ as shown in the rightmost diagram of Fig. 14. The labels on the tetrahedron are permuted from (i, j, k) to (j, k, i) because we wish to retain the polynomials M_{2j} and M_{3j} and replace the polynomials L_{1j} by M_{1j}. Now the labels $g_{k,\alpha}$ are computed as follows: For $|\alpha| = 0, 1$, let $i = 2 - \alpha$; then

$$L_{1i} = g_{1,\alpha} M_{1,\alpha_1+1} + g_{2,\alpha} M_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}.$$

Again in our special case these labels are the same as in the first tetrahedron and are shown in the right diagram of Fig. 14. After the computation is complete, the new coefficients U_α emerge on the nodes on the base triangle. These new coefficients are as follows: $U_{200} = \frac{1}{2}T_{200}$, $U_{110} = -\frac{1}{2}T_{200} + T_{110}$, $U_{020} = T_{020}$, $U_{101} = -\frac{1}{2}T_{200} + T_{101}$, $U_{011} = T_{011}$, $U_{002} = T_{002}$.

These coefficients express the polynomial L with respect to the L-basis defined by the knot-net $\mathscr{M} = \{\{M_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, 2\}$. The change of basis algorithm is now complete. In terms of the initial coefficients R_α, the final coefficients U_α, are: $U_{200} = \frac{1}{2}R_{200}$, $U_{110} = -\frac{1}{2}R_{200} - \frac{1}{2}R_{020} + R_{110}$, $U_{020} = \frac{1}{2}R_{020}$, $U_{101} = -\frac{1}{2}R_{200} - \frac{1}{2}R_{002} + R_{101}$, $U_{011} = -\frac{1}{2}R_{020} - \frac{1}{2}R_{002} + R_{011}$, $U_{002} = \frac{1}{2}R_{002}$.

The general change of basis algorithm from any L-basis to any other L-basis is obtained by following essentially the same procedure. Suppose we are given the coefficients R_α of a polynomial L of degree n with respect to an L-basis $\{l^n_\alpha\}$ defined by the knot-net $\mathscr{L} = \{\{L_{1j}\}, \{L_{2j}\}, \{L_{3j}\}, j = 1, ..., n\}$. We would like to compute the coefficients U_α of this polynomial L with respect to another L-basis $\{p^n_\alpha\}$ defined by another knot-net $\mathscr{M} = \{\{M_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, ..., n\}$.

The general change of basis algorithm is constructed in the following manner:

1. Build three tetrahedra. For each tetrahedron, $(n+1-i)(n+2-i)/2$ nodes are placed at the ith level of the tetrahedron for $i = 0, ..., n$. The labels $g_{k,\alpha}$ along the edges of the first tetrahedron are computed for $|\alpha| = 0, ..., n-1$, from

$$L_{3i} = g_{1,\alpha} L_{1,\alpha_1+1} + g_{2,\alpha} L_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}, \qquad i = n - |\alpha|.$$

The labels for the second and the third tetrahedron are computed in a similar fashion. We assume that the intermediate knot-nets $\{\{L_{1j}\}, \{L_{2j}\}, \{M_{3j}\}, j = 1, ..., n\}$ are linearly independent.

2. Point the arrows on the tetrahedron downwards and place the original coefficients R_α along the lateral face of the pyramid. Carry out the computation and collect the new coefficients S_α along the base of the pyramid.

3. Repeat steps 1 and 2 twice with the second and third tetrahedron using the output of the previous step as the input of the next step. After three steps, the coefficients at the base of the tetrahedron are the desired coefficients U_α.

We can use this general change of basis algorithm for L-bases to convert from Bernstein–Bézier to Lagrange form. Since the Lagrange coefficients are the values of the bivariate polynomial at $O(n^2)$ nodes and since this change of basis algorithm is $O(n^3)$, converting from Bernstein–Bézier to Lagrange form evaluates the polynomial at $O(n^2)$ points with an amortized cost of $O(n)$ computations per point. This cost compares very favorably with the de Casteljau evaluation algorithm for Bernstein–Bézier surfaces which costs $O(n^3)$ computations per point.

Finally, the transformation between a B-basis and an L-basis can be achieved by factoring through the Bernstein–Bézier or multinomial bases, which are both B-bases and L-bases. For example, given a polynomial with respect to a power B-basis one can convert from the power B-basis to either a multinomial or Bernstein–Bézier basis using the change of basis algorithms between B-bases [LG95a] and then convert from the multinomial or Bernstein–Bézier basis to the desired L-basis, say a Lagrange L-basis, by using the change of basis algorithms between L-bases described above. Again when the L-basis is a Lagrange basis, this change of basis algorithm evaluates the B-patch at $O(n^2)$ points with an amortized cost of $O(n)$ computations per point. This compares favorably with the generalized de Boor evaluation algorithm for B-patches which requires $O(n^3)$ computations per point.

6. CONCLUSIONS AND FUTURE WORK

Lagrange and Newton bases play a very important role in point and derivative interpolation problems for surfaces. We have demonstrated that a very interesting subclass of bivariate Lagrange bases can be realized as bivariate L-bases. We have also demonstrated that a very interesting subclass of bivariate Newton bases can be realized as bivariate L-bases. Using the principle of duality between L-bases and B-bases, we have established that Lagrange L-bases and generalized Newton L-bases are dual respectively to power B-bases and generalized Newton dual B-bases. We have also discussed the duality between Bernstein–Bézier and multinomial bases, which arise as special cases of both B-bases and L-bases. We went on to provide a geometric interpretation of the duality principle as point-line duality, where a point or a vector in a B-basis corresponds to a line in an L-basis. This duality provides strong geometric insight for working with these bases. We have also presented a rich collection of lattices that admit bivariate Lagrange and Newton L-bases, which solve uniquely certain well-studied point and derivative interpolation problems in several variables.

We have presented a unified collection of change of basis algorithms based on the principle of duality for a wide variety of polynomial bases used in representing surfaces including the Bernstein–Bézier, multinomial, Lagrange, power, Newton, Newton dual, B-bases, and L-bases. We have also given an example of change of basis from Lagrange to Bernstein–Bézier form.

This research has opened up several interesting new questions. The generalization of the de Boor evaluation algorithm for bivariate B-bases is well known. The dual evaluation algorithm for bivariate L-bases was described by the authors in an earlier work [LG95a]. Does this algorithm yield new procedures for the evaluation of multinomial, Lagrange, and Newton bases? The de Casteljau subdivision algorithm for a Bernstein–Bézier patch is very well known. What is the corresponding dual algorithm? We plan to investigate the dual evaluation algorithms for L-patches [LG95c], dual de Casteljau subdivision algorithm for Bernstein–Bézier surfaces [LG95b], and duality between degree elevation and differentiation formulas [LG94a] in forthcoming papers. Although we have discussed point-line duality, we have observed that this duality is not self-dual. It would be interesting to explore self-duality and discover new computational algorithms based on self-duality. It would also be worthwhile to extend the notions of B-bases and L-bases to enlarge the configurations of points and lines for which Lagrange or Newton bases exist but for which a Lagrange L-basis or Newton L-basis does not exist.

APPENDIX

Let $\{\mathbf{u}_{1j}, \mathbf{u}_{2j}, \mathbf{u}_{3j}, j=1, ..., n\}$ be a knot-net of vectors. Suppose the vectors $(\mathbf{u}_{1, \alpha_1+1}, \mathbf{u}_{2, \alpha_2+1}, \mathbf{u}_{3, \alpha_3+1})$ are linearly dependent for $|\alpha| = n$, $0 \leqslant \alpha_k \leqslant n-1$. This appendix gives an inductive proof of the fact that up to constant multiples each element of the corresponding B-basis b^n_α is an nth power of a linear polynomial. The inductive proof is very interesting in its own right and reveals the underlying structure of the recurrence diagram. Also the technique used in the proof is valuable in other situations, including the proof of the generalized Aitken–Neville algorithm for Lagrange L-bases [LG95c].

Let q_α be the equations of the lines associated with the power B-basis as defined in Section 4.2.2.

THEOREM 2.

$$b^n_\alpha(\mathbf{u}) = \frac{(q_\alpha)^n}{\prod_{j=1, ..., \alpha_i;\, i=1, 2, 3} q_\alpha(\mathbf{u}_{ij})}.$$

Proof. The proof is by induction. The case $n=1$ reduces to the simple case of a triangle defined by the points $\mathbf{u}_{11}$, $\mathbf{u}_{21}$, and $\mathbf{u}_{31}$. The lines opposite to these vertices are Q_{100}, Q_{010}, and Q_{001}, respectively In this case it can readily be verified that the basis functions b^1_{100}, b^1_{010}, and b^1_{001} are $q_{100}/q_{100}(\mathbf{u}_{11})$, $q_{010}/q_{010}(\mathbf{u}_{21})$, and $q_{001}/q_{001}(\mathbf{u}_{31})$, respectively. If the three points lie in the affine plane, then these basis functions are simply the barycentric coordinates with respect to the triangle formed by $\mathbf{u}_{11}$, $\mathbf{u}_{21}$, and $\mathbf{u}_{31}$.

The inductive hypothesis assumes that the statement of the theorem is true for $n-1$. We now prove that the statement holds for n. To this end, choose an arbitrary but fixed α. Recall that b^n_α is defined from the recurrence in Eq. (2) by substituting $C_\alpha = 1$ and setting the other constants $C_\beta = 0$. With this choice of constants, at the highest level of the recurrence when $|\alpha| = 0$, we make the following observations:

1. $C^n_0(\mathbf{u}) = b^n_\alpha$.

2. $h_{k,000}(\mathbf{u})$ are the coordinates of $\mathbf{u}$ with respect to the triangle $(\mathbf{u}_{11}, \mathbf{u}_{21}, \mathbf{u}_{31})$, that is, $h_{1,000}(\mathbf{u}) = q_{100}(\mathbf{u})/q_{100}(\mathbf{u}_{11})$, $h_{2,000}(\mathbf{u}) = q_{010}(\mathbf{u})/q_{010}(\mathbf{u}_{21})$, and $h_{3,000}(\mathbf{u}) = q_{001}(\mathbf{u})/q_{001}(\mathbf{u}_{31})$.

3. Finally, $C^{n-1}_{e_k}(\mathbf{u})$ for $k = 1, 2, 3$ are obtained by running only $n-1$ levels of the recurrence in Eq. (2), and therefore, $C^{n-1}_{e_k}(\mathbf{u}) = b^{n-1}_{\alpha - e_k}(\mathbf{u})$, where $b^{n-1}_{\alpha - e_k}(\mathbf{u})$ are the B-basis functions corresponding to the knot-nets $\mathscr{W}_1 = \{(\hat{\mathbf{u}}_{11}, ..., \mathbf{u}_{1n}), (\mathbf{u}_{21}, ..., \hat{\mathbf{u}}_{2n}), (\mathbf{u}_{31}, ..., \hat{\mathbf{u}}_{3n})\}$, $\mathscr{W}_2 = \{(\mathbf{u}_{11}, ..., \hat{\mathbf{u}}_{1n}), (\hat{\mathbf{u}}_{21}, ..., \mathbf{u}_{2n}), (\mathbf{u}_{31}, ..., \hat{\mathbf{u}}_{3n})\}$, and $\mathscr{W}_3 = \{(\mathbf{u}_{11}, ..., \hat{\mathbf{u}}_{1n}), (\mathbf{u}_{21}, ..., \hat{\mathbf{u}}_{2n}), (\hat{\mathbf{u}}_{31}, ..., \mathbf{u}_{3n})\}$, respectively, where $\hat{\mathbf{u}}$ means that the term $\mathbf{u}$ is missing.

Putting these observations together, Eq. (2) at the highest level of recurrence when $|\alpha| = 0$ now translates into the following:

$$b^n_\alpha(\mathbf{u}) = \frac{q_{100}(\mathbf{u})}{q_{100}(\mathbf{u}_{11})}\, b^{n-1}_{\alpha - e_1}(\mathbf{u}) + \frac{q_{010}(\mathbf{u})}{q_{010}(\mathbf{u}_{21})}\, b^{n-1}_{\alpha - e_2}(\mathbf{u}) + \frac{q_{001}(\mathbf{u})}{q_{001}(\mathbf{u}_{31})}\, b^{n-1}_{\alpha - e_3}(\mathbf{u}). \quad (3)$$

We now prove that the knot-nets $\mathscr{W}_1$, $\mathscr{W}_2$, and $\mathscr{W}_3$ satisfy the linear independence condition so that they actually are knot-nets and that they also satisfy the linear dependence condition of the power basis.

To this end, let us denote the knot-net of $\mathscr{W}_1$ also as follows: $\{\mathbf{w}_{1j}, \mathbf{w}_{2j}, \mathbf{w}_{3j}, j = 1, ..., 3\}$. The knot-net $\mathscr{W}_1$ satisfies the linear independence condition because $(\mathbf{w}_{1,\beta_1+1}, \mathbf{w}_{2,\beta_2+1}, \mathbf{w}_{3,\beta_3+1})$ is linearly independent for $0 \leqslant |\beta| \leqslant n-2$ iff $(\mathbf{u}_{1,\beta_1+1}, \mathbf{u}_{2,\beta_2+1}, \mathbf{u}_{3,\beta_3+1})$ is linearly independent for $0 \leqslant |\beta| \leqslant n-2$. The latter condition is, however, equivalent to the linear independence of $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ where $\alpha = (\beta_1 + 1, \beta_2, \beta_3)$ with $1 \leqslant |\alpha| \leqslant n-1$, which is satisfied because of the linear independence condition on the original knot-net $\mathscr{U}$. Similarly, the knot-nets $\mathscr{W}_2$ and $\mathscr{W}_3$ are linearly independent.

Moreover, the knot-net $\mathscr{W}_1$ satisfies the linear dependence condition of the power basis because $(\mathbf{w}_{1,\beta_1+1}, \mathbf{w}_{2,\beta_2+1}, \mathbf{w}_{3,\beta_3+1})$ are linearly dependent for $|\beta| = n-1, 0 \leqslant \beta_k \leqslant n-2$ iff $(\mathbf{u}_{1,\beta_1+1}, \mathbf{u}_{2,\beta_2+1}, \mathbf{u}_{3,\beta_3+1})$ are linearly dependent for $|\beta| = n-1$, $0 \leqslant \beta_k \leqslant n-2$. The latter condition is, however, equivalent to the linear dependence of $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ where $\alpha = (\beta_1 + 1, \beta_2, \beta_3)$ with $|\alpha| = n$, $0 \leqslant \alpha_k \leqslant n-1$, which is satisfied because of the linear dependence condition on the original knot-net $\mathscr{U}$. Similarly, the knot-nets $\mathscr{W}_2$ and $\mathscr{W}_3$ satisfy the linear dependence condition of the power basis.

Since the knot-nets $\mathscr{W}_1$, $\mathscr{W}_2$, and $\mathscr{W}_3$ satisfy the assumptions of the theorem, we can apply the inductive hypothesis to these knot-nets. Now observe that the line corresponding to the power B-basis $b^{n-1}_{\alpha-e_1}$ with $|\alpha|=n$ is the line determined by $\mathbf{w}_{1,\alpha_1}$, $\mathbf{w}_{2,\alpha_2+1}$, and $\mathbf{w}_{3,\alpha_3+1}$, which in turn is the line determined by $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$, and is therefore q_α. Similar assertions hold for $b^{n-1}_{\alpha-e_2}$ and $b^{n-1}_{\alpha-e_3}$. Hence the inductive hypothesis yields:

$$b^{n-1}_{\alpha-e_k}(\mathbf{u}) = \frac{(q_\alpha)(\mathbf{u}_{k1})}{\prod_{j=1,\ldots,\alpha_i;\, i=1,2,3} q_\alpha(\mathbf{u}_{ij})} (q_\alpha(\mathbf{u}))^{n-1}.$$

Substituting this formula into Eq. (3), we obtain:

$$b^n_\alpha(\mathbf{u}) = \frac{(q_\alpha(\mathbf{u}))^{n-1}}{\prod_{j=1,\ldots,\alpha_i;\, i=1,2,3} q_\alpha(\mathbf{u}_{ij})} \left(\frac{q_\alpha(\mathbf{u}_{11})}{q_{100}(\mathbf{u}_{11})} q_{100}(\mathbf{u}) + \frac{q_\alpha(\mathbf{u}_{21})}{q_{010}(\mathbf{u}_{21})} q_{010}(\mathbf{u}) + \frac{q_\alpha(\mathbf{u}_{31})}{q_{001}(\mathbf{u}_{31})} q_{001}(\mathbf{u}) \right). \tag{4}$$

Now observe that the expression I within the brackets in Eq. (4) is a linear polynomial and is therefore completely determined by its value at three independent points. However, since $I(\mathbf{u}_{k1}) = q_\alpha(\mathbf{u}_{k1})$ for $k=1, 2, 3$ and $\mathbf{u}_{11}$, $\mathbf{u}_{21}$, and $\mathbf{u}_{31}$ are linearly independent points, it follows that $I = q_\alpha(\mathbf{u})$. Thus the statement of the theorem is established. ∎

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